Evaluation of the Euclidean action

$$S_E[A] = \int d^4x_E \; \frac{1}{4}\left(F^a_{\mu\nu} F^a_{\mu\nu}\right)_E$$

Write this in terms of Ansatz function $f(x^2)$

Previously calculated:

$$F^a_{\mu\nu} = \frac{-4}{g}\left[\frac{x^2 f' - f(1-f)}{x^4}\left(\eta^a{}_{\nu\rho} x_\rho x_\mu - \eta^a{}_{\mu\rho} x_\rho x_\nu\right) - \frac{f(1-f)}{x^2}\eta^a{}_{\mu\nu}\right]$$

So, squaring:

$$F^a_{\mu\nu} F^a_{\mu\nu} = \frac{16}{g^2}\left[\mathrm{I} + \mathrm{II}\right]\left[A + B\right]$$

Simplify tensor structures:

IA: $\left(\eta^a{}_{\nu\rho} x_\rho x_\mu - \eta^a{}_{\mu\rho} x_\rho x_\nu\right)\left(\eta^a{}_{\nu\sigma} x_\sigma x_\mu - (\mu \leftrightarrow \nu)\right)$

$$= \left(\eta^a{}_{\nu\rho} x_\rho x_\mu \eta^a{}_{\nu\sigma} x_\sigma x_\mu - \eta^a{}_{\mu\rho} x_\rho x_\nu \eta^a{}_{\nu\sigma} x_\sigma x_\mu\right) \times 2$$

vanishes by antisymmetry.

$$= 2\underbrace{\eta^a{}_{\nu\rho}\eta^a{}_{\nu\sigma}}_{3\delta_{\rho\sigma}} x^2 x_\sigma x_\rho$$

(using identity for 't Hooft symbols)

$$= 6(x^2)^2$$

IIA + IB: $2\left(\eta^a{}_{\nu\rho} x_\rho x_\mu - \eta^a{}_{\mu\rho} x_\rho x_\nu\right)\eta^a{}_{\mu\nu}$

$$= 2\left(\underbrace{\eta^a{}_{\nu\rho}\eta^a{}_{\mu\nu}} x_\rho x_\mu - \underbrace{\eta^a{}_{\mu\rho}\eta^a{}_{\mu\nu}} x_\rho x_\nu\right)$$

(identity)

$$= 2\left(-3\delta_{\rho\mu} x_\rho x_\mu - 3\delta_{\rho\nu} x_\rho x_\nu\right)$$

$$= -12x^2$$

IIB: $\eta^a{}_{\mu\nu}\eta^a{}_{\mu\nu} = 12$ (identity)

So, together: the kinetic term is:

$$\frac{1}{4}\left(F^a_{\mu\nu}F^a_{\mu\nu}\right)_E = \frac{1}{4}\frac{16}{g^2}\left[\left(\frac{x^2f'-f(1-f)}{x^4}\right)^2 6(x^2)^2 - \frac{x^2f'-f(1-f)}{x^4}\frac{f(1-f)}{x^2}(-12x^2) + \left(\frac{f(1-f)}{x^2}\right)12\right]$$

$$= \frac{24}{g^2}\frac{1}{(x^2)^2}\left[x^4(f')^2 + f^2(1-f)^2\right] \qquad \underline{\text{NOTE}}\ f' = \frac{df}{dx^2}$$

So that the Euclidean action for the instanton is:

$$S_E[A_{inst}] = \int d^4x_E \frac{24}{g^2}\frac{1}{(x^2)^2}\left[x^4(f')^2 + f^2(1-f)^2\right]$$

convert to hyper-spherical polars:

$$= \underbrace{\int d\Omega_4}_{2\pi^2}\int_0^\infty dr\, r^3 \frac{24}{g^2}\frac{1}{r^4}\left[r^4(f')^2 + f^2(1-f)^2\right] \qquad \text{and } f' = \frac{df}{dr^2}$$

factor

$$= \frac{2\pi^2}{g^2}\int_0^\infty dr\,\frac{24}{r}\left[r^4(f')^2 + f^2(1-f)^2\right]$$

Plug in $f(r^2) = \frac{r^2}{r^2+\rho^2}$ as derived from solving Euclidean EOM:

$$f'(r^2) = \frac{\rho^2}{(r^2+\rho^2)^2}$$

$$S_E[A_{inst}] = \frac{2\pi^2}{g^2}\int_0^\infty dr\,\frac{24}{r}\left[r^4\left(\frac{\rho^2}{(r^2+\rho^2)^2}\right)^2 + \left(\frac{r^2}{r^2+\rho^2}\right)^2\left(1-\frac{r^2}{r^2+\rho^2}\right)^2\right]$$

$$= \frac{2\pi^2}{g^2}\int_0^\infty dr\,\frac{48r^3\rho^4}{(r^2+\rho^2)^4}$$

$$= \frac{2\pi^2}{g^2}\left[\frac{-4\rho^4(3r^2+\rho^2)}{(r^2+\rho^2)^3}\right]_0^\infty = \frac{2\pi^2}{g^2}\left[-0-(-4)\right] = \frac{8\pi^2}{g^2}$$

$$\boxed{S_E[A_{inst}] = \frac{8\pi^2}{g^2}}$$

Evaluation of winding number

$$\nu = \frac{g^2}{16\pi^2} \int d^4x_E \ \frac{1}{2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}$$

instanton solution:

$$F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu} = \frac{4}{g} \frac{\rho^2}{(x^2+\rho^2)^2} \eta^a{}_{\mu\nu}$$

$$\therefore F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} = \frac{16}{g^2} \left[\frac{\rho^2}{(x^2+\rho^2)^2} \right] \underbrace{\eta^a{}_{\mu\nu} \eta^a{}_{\mu\nu}}_{12}$$

$$= \frac{16 \times 12}{g^2} \frac{\rho^4}{(x^2+\rho^2)^4}$$

$$\nu = \frac{g^2}{16\pi^2} \frac{1}{2} \int d^4x_E \ \frac{16 \times 12}{g^2} \frac{\rho^4}{(x^2+\rho^2)^4}$$

$$= \frac{\cancel{g^2}}{\cancel{16}\pi^2} \frac{\cancel{16} \times \cancel{12}^{\,6}}{\cancel{g^2}} \frac{1}{\cancel{2}} \underbrace{\int d\Omega_4}_{2\pi^2} \int dr \ r^3 \frac{\rho^4}{(r^2+\rho^2)^4}$$

$$= \frac{6}{\pi^2} \times 2\pi^2 \int dr \ r^3 \frac{\rho^4}{(r^2+\rho^2)^4}$$

ch var: $r = \rho\bar{r}$ $\quad \bar{r}$ = dimensionless

$$= 12 \int_0^\infty d\bar{r} \ \frac{\bar{r}^3}{(\bar{r}^2+1)^4} = 1$$