Visualizing the Yang-Mills instanton solution

The physical fields are the non-abelian analogs of the electric, $\vec{E}^a$ and magnetic $\vec{B}^a$ fields, since they transform homogenously.

(or antiself-dual)

The instanton solution is self-dual, so sufficient to plot only $\vec{E}^a$ or $\vec{B}^a$.

$$\rightarrow \text{Plot } \vec{B}^a = \overset{①}{\vec{\nabla}\times\vec{A}^a} - \overset{②}{\frac{g}{2}}\epsilon^{abc}\vec{A}^b\times\vec{A}^c$$

Need $A^a_i = \dfrac{-2}{g}\dfrac{\eta^a{}_{i\nu}x_\nu}{x^2+\rho^2} = \dfrac{-2}{g}\dfrac{x_\nu}{x^2+\rho^2}\left(\epsilon^a{}_{i\nu}+\delta^a{}_i\delta_{\nu 0}\right)$

$$= \frac{-2}{g}\frac{1}{\tau^2+\vec{x}^2+\rho^2}\left(\epsilon^a{}_{ij}x_j + \delta^a{}_i\tau\right)$$

$\tau \equiv$ Eucl. time.

Start with ①:

$$(\vec{\nabla}\times\vec{A}^a)_i = \epsilon_{ijk}\,\partial_j\left(\frac{-2}{g}\frac{1}{\tau^2+\vec{x}^2+\rho^2}\left(\epsilon^a{}_{kl}x_l + \delta^a{}_k\tau\right)\right)$$

$$= \frac{-2}{g}\partial_j\left(\frac{1}{\tau^2+\vec{x}^2+\rho^2}\left(\underbrace{\epsilon_{ijk}\epsilon^a{}_{kl}}_{\delta_{il}\delta^a_j-\delta^a_i\delta_{jl}}x_l + \epsilon_{ijk}\delta^a{}_k\tau\right)\right)$$

$$= \frac{-2}{g}\partial_j\left(\frac{1}{\tau^2+\vec{x}^2+\rho^2}\left(x_i\delta^a{}_j - x_j\delta^a{}_i + \epsilon^a{}_{ij}\tau\right)\right)$$

Now, putting $f(\vec{x}^2) = \tau^2+\vec{x}^2+\rho^2$,

we have $\partial_j\dfrac{1}{f(\vec{x}^2)} = -f(\vec{x}^2)^{-2}f'(\vec{x}^2)\,2x_j = \dfrac{-2x_j}{(\tau^2+\vec{x}^2+\rho^2)^2}$

$$(\vec{\nabla}\times\vec{A}^a)_i = \frac{-2}{g}\left[\frac{-2x_j}{(\tau^2+\vec{x}^2+\rho^2)^2}\left(x_i\delta^a{}_j - x_j\delta^a{}_i + \epsilon^a{}_{ij}\tau\right) + \frac{1}{(\tau^2+\vec{x}^2+\rho^2)}\left(\delta^a{}_i - 3\delta^a{}_i + 0\right)\right]$$

Factor out $\dfrac{-2}{(\tau^2+\vec{x}^2+\rho^2)^2}$

cancel

$$(\vec\nabla\times\vec A^a)_i = \frac{+4}{g(\tau^2+\vec x^2+\rho^2)^2}\Big[\, x_i x^a - \vec x^2\delta^a{}_i + \epsilon^a{}_{ij}x_j\tau - \frac{1}{2}(\tau^2+\vec x^2+\rho^2)(-2\delta^a{}_i$$

$$= \frac{4}{g(\tau^2+\vec x^2+\rho^2)^2}\Big[\, x_i x^a + \epsilon^a{}_{ij}x_j\tau + (\tau^2+\rho^2)\delta^a{}_i\Big]$$

Now calculate ②:

$$-\frac{g}{2}\epsilon^{abc}\vec A^b\times\vec A^c$$

$$= -\frac{g}{2}\epsilon^{abc}\left(\frac{-2}{g}\frac{1}{\tau^2+\vec x^2+\rho^2}\right)^2\epsilon_{ijk}\left(\epsilon^b{}_{jl}x_l+\delta^b{}_j\tau\right)\left(\epsilon^c{}_{km}x_m+\delta^c{}_k\tau\right)$$

$$= \frac{-2}{g}\frac{1}{(\tau^2+\vec x^2+\rho^2)^2}\Big[\underset{\mathrm{I}}{\epsilon^{abc}\epsilon_{ijk}\epsilon^b{}_{jl}\epsilon^c{}_{km}x_l x_m} + \underset{\mathrm{II}}{\epsilon^{abc}\epsilon_{ijk}\epsilon^b{}_{jl}\delta^c{}_k x_l\tau}$$

$$+ \underset{\mathrm{III}}{\epsilon^{abc}\epsilon_{ijk}\epsilon^c{}_{km}\delta^b{}_j x_m\tau} + \underset{\mathrm{IV}}{\epsilon^{abc}\epsilon_{ijk}\delta^b{}_j\delta^c{}_k\tau^2}\Big]$$

Simplify the four tensor structures:

I $$\epsilon^{abc}\underbrace{\epsilon_{ijk}\epsilon^a{}_{jl}}_{\delta^b{}_i\delta_{kl}-\delta^b{}_k\delta_{il}}\epsilon^c{}_{km}x_l x_m = \left(\delta^b{}_i\delta_{kl}-\delta^b{}_k\delta_{il}\right)\underbrace{\epsilon^{abc}\epsilon^c{}_{km}}_{\delta^a{}_k\delta^b{}_m-\delta^a{}_m\delta^b{}_k}x_l x_m$$

$$= \left(\delta_{im}\delta^a{}_l - \delta^a{}_m\delta_{il} - \delta_{il}\delta^a{}_m + 3\delta_{il}\delta^a{}_m\right)x_l x_m$$

$$= \left(\delta_{im}\delta^a{}_l + \delta_{il}\delta^a{}_m\right)x_l x_m = x_i x^a + x_i x^a = 2x_i x^a$$

II $$\epsilon^{abc}\epsilon_{ijk}\epsilon^b{}_{jk}\delta^c{}_k x_l\tau = \underbrace{\epsilon^{abc}\epsilon^c{}_{ij}}_{\delta^a{}_i\delta^b{}_j-\delta^a{}_j\delta^b{}_i}\epsilon^b{}_{jk}x_l\tau = -\delta^a{}_j\epsilon_{ijl}x_l\tau$$

vanishes by symm.

$$= +\epsilon^a{}_{ij}x_j\tau$$

(III) $\epsilon^{abc}\epsilon_{ijk}\epsilon^{c}{}_{km}\delta^{b}{}_{j}\,x_m\tau = \left(\delta^a{}_k\delta^b{}_m - \delta^a{}_m\delta^b{}_k\right)\epsilon_{ijk}\delta^b{}_j\,x_m\tau$

$\delta^a{}_k\delta^b{}_m - \delta^a{}_m\delta^b{}_k$

$$= \epsilon^a{}_{ij}\delta_{mj}x_m\tau - \underbrace{\epsilon^b{}_{ij}\delta^b{}_j x^a\tau}_{\text{vanishes by symm}} = \epsilon^a{}_{ij}x_j\tau$$

(IV) $\epsilon^{abc}\epsilon_{ijk}\delta^b{}_j\delta^c{}_k\tau^2 = \epsilon^a{}_{jk}\epsilon_{ijk}\tau^2 = 2\delta^a{}_i\tau^2$

So,

$$-\frac{g}{2}\epsilon^{abc}(\vec{A}^b\times\vec{A}^c)_i = \frac{-2}{g}\frac{1}{(\tau^2+\vec{x}^2+\rho^2)^2}\left(2x_ix^a + \epsilon^a{}_{ij}x_j\tau + \epsilon^a{}_{ij}x_j\tau + 2\delta^a{}_i\tau^2\right)$$

$$= \frac{-4}{g}\frac{1}{(\tau^2+\vec{x}^2+\rho^2)^2}\left(x_ix^a + \epsilon^a{}_{ij}x_j\tau + \delta^a{}_i\tau^2\right)$$

So, together, the non-abelian magnetic field, B^a_i is

$$B^a_i = \frac{4}{g}\frac{1}{(\tau^2+\vec{x}^2+\rho^2)^2}\Big[\cancel{x_ix^a} + \cancel{\epsilon^a{}_{ij}x_j\tau} + (\cancel{\tau^2}+\rho^2)\delta^a{}_i - \cancel{x_ix^a} - \cancel{\epsilon^a{}_{ij}x_j\tau} - \cancel{\delta^a{}_i\tau^2}\Big]$$

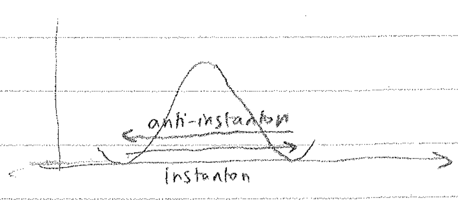
$$\boxed{(B^a_i)^{inst} = \frac{4}{g}\frac{\rho^2}{(\tau^2+\vec{x}^2+\rho^2)^2}\delta^a{}_i} \equiv (E^a_i)^{inst}$$

For anti-instantons, $(B^a_i)^{anti} = -(E^a_i)^{anti}$

Internal isospin index correlated with spatial index.

Since $B^a_i \xrightarrow{T} -B^a_i$
$E^a_i \xrightarrow{T} +E^a_i$,

$T \equiv$ imaginary time reversal.



instanton $\xleftrightarrow{T}$ anti-instanton