Scaling Arguments against the existence of static classical solutions (Derrick's theorem)

Scale transformations (in d Euclidean dimensions)

Consider the general Euclideanized Lagrangian of N scalar fields. $(i,j=1,2\ldots N)$

$$\mathcal{L}_E = \tfrac{1}{2} F^{ij}\, \partial_\mu \phi^i \partial_\mu \phi^j + V(\phi) \qquad F^{ij} \equiv F^{ij}(\phi)$$

Suppose that $\phi_c^i(\vec{x})$ is a <u>static</u> solution of classical field equations with finite energy. (solitons and instantons <u>only</u>; not bounce or sphaleron).

$$E[\phi] = \int d^d x_E \left[\frac{1}{2} G^{ij} (\vec\nabla \phi^i).(\vec\nabla\phi^j) + V(\phi)\right] \qquad G^{ij} = G^{ij}(\phi)$$

If G^{ij} is a positive-definite matrix, and $V(\phi)$ is bounded from below (with $V(\phi)=0$ at classical minima), then $E[\phi] \geq 0$ for all field configurations.

Consider a family of field configurations, parametrized by a variational parameter λ which characterizes the "size" of the solution in Euclidean space:

$$\phi_c(x) \longrightarrow \phi_c(\lambda x) \qquad \underset{\text{inverse-length}}{\lambda}:\ \underset{\substack{\uparrow \\ \text{BIG} \\ \text{SOLUTION}}}{0} \longrightarrow \underset{\substack{\uparrow \\ \text{TINY} \\ \text{SOLUTION}}}{\infty}$$

$$E[\phi_\lambda] = \int d^d x_E \left[\frac{1}{2} S^{ij} (\vec\nabla \phi_\lambda^i).(\vec\nabla\phi_\lambda^j) + V(\phi_\lambda)\right]$$

ch. var $\vec{y} = \lambda \vec{x} \Rightarrow d\vec{x} = \frac{1}{\lambda} d\vec{y}$ and $\vec\nabla_x = \lambda \vec\nabla_y$

$$= \frac{1}{\lambda^d} \int d^d y_E \left[\frac{1}{2} S^{ij} \lambda^2 (\vec\nabla_y \phi^i_{(y)}).(\vec\nabla_y \phi^j_{(y)}) + V(\phi_{(y)})\right]$$

$$= \frac{\lambda^2}{\lambda^d} \underset{\uparrow}{KE[\phi_c]} + \frac{1}{\lambda^d} \underset{\uparrow}{PE[\phi_c]}\,, \qquad \begin{array}{l} KE = \text{kinetic energy} \\ PE = \text{Potential energy} \end{array}$$

These energies are independent of variational parameter.
All dependence of $E[\phi_\lambda]$ on λ is shown explicitly.

Seek optimized λ by extremizing $E[\phi_\lambda]$.

$$\frac{\partial E}{\partial \lambda} = (2-d)\frac{1}{\lambda^{d-1}} KE[\phi_c] - d\,\frac{1}{\lambda^{-d+1}} PE[\phi_c] = 0$$

Multiply through by λ^{-d+1}

$$= (2-d)\,\lambda^2\, KE[\phi_c] - d\; PE[\phi_c] = 0 .$$

$$\Rightarrow \quad \lambda_{opt} = \sqrt{\frac{d}{2-d}\,\frac{PE[\phi_c]}{KE[\phi_c]}}$$

Existence of λ_{opt} in various Euclidean dimensions

Euclidean dimension d	λ_{opt}	
0	undef.	because $KE[\phi_c]=0$.
1	$(PE/KE)^{1/2}$	✓
2	∞	
3	imaginary	
4	imaginary	
⋮	⋮	

Non-trivial static solutions of field equations are possible only in 1D.
(QM instantons, 1+1 D "kink" soliton). — must modify theory to obtain solutions in other dimensions.

! **Note:** This does not apply to "bounce" and sphaleron solutions since these solutions are not in stable equilibrium — fluctuation spectrum contains a negative mode.

A more transparent way of understanding Derrick's theorem is by investigating how KE and PE scale with λ. A stable solution exists if there is competition between the two energies.

Derreck's theorem – argument by competing energies (Viral theorem)

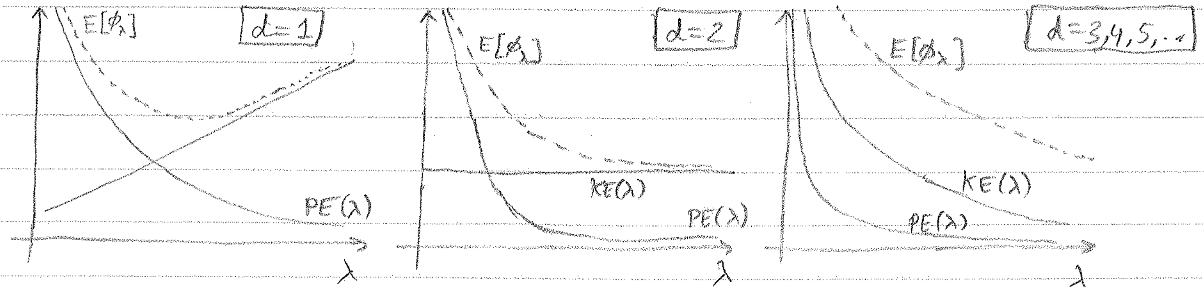
Scaler field theory in d - Euclidean dimensions

$$E[\phi] = \int d^d x_E \left[\frac{1}{2} G^{ij} (\vec{\nabla}\phi^i)\cdot(\vec{\nabla}\phi^j) + V(\phi) \right]$$

Return to energy as function of variational parameter λ:

$$E[\phi_\lambda] = \underbrace{\frac{\lambda^2}{\lambda^d} KE[\phi_c]}_{KE[\phi_c;\lambda]} + \underbrace{\frac{1}{\lambda^d} PE[\phi_c]}_{PE[\phi_c;\lambda]} .$$

We can now make plots of the kinetic energy and potential energy as a function of λ



Notice, in $d=1$ dimensions, PE favors tiny solution (λ big), KE favors big solution (λ small); hence, there is static equilibrium between KE and PE. In $d>1$ dimensions, $\lambda \to \infty$ meaning solution dissappears.

To see this easily mathematically, note that KE(λ) and PE(λ) are monotonic in their variables. Therefore, it is sufficient to check the sign of their slopes at an arbitrary point — say $\lambda = 1$.

$$\frac{\partial E}{\partial \lambda} = (2-d)\frac{1}{\lambda^{d-1}} KE[\phi_c] - d \frac{1}{\lambda^{-d+1}} PE[\phi_c]$$

$$\xrightarrow{\lambda=1} \underbrace{(2-d)\, KE[\phi_c]}_{\left.\frac{d(KE)}{d\lambda}\right|_{\lambda=1}} \quad \underbrace{-d\; PE[\phi_c]}_{\left.\frac{d(PE)}{d\lambda}\right|_{\lambda=1}}$$

$d=1$	+	−	← competition!
$=2$	0	−	
$=3,4,\ldots$	−	−	