Schwinger's Proper-Time Representation

Want to compute functional determinants – 1 loop Euclidean effective action:

$$\Gamma_{eff}[\phi_c] = \Gamma_{tree}[\phi] \pm \frac{1}{2} \ln \det\left(\frac{\mathcal{O}[\phi]}{\mathcal{O}[0]}\right) \qquad \begin{matrix} + \text{ for fermionic} \\ - \text{ for bosonic} \end{matrix}$$

Notice the identity:

$$-\int_0^\infty dt\, \frac{1}{t}\left(e^{-ta} - e^{-tb}\right) = \ln\left(\frac{a}{b}\right) \qquad [\text{valid for } a>0,\ b>0]$$

Proof: Regulate the integrand ($z \to 0$ taken afterward)

$$-\int_0^\infty dt\, \frac{1}{t}\left(e^{-ta} - e^{-tb}\right) = \lim_{z\to 0} -\int_0^\infty dt\, t^{z-1}\left(e^{-ta} - e^{-tb}\right)$$

In the first term, rescale integration variable $t \to t/a$

In " second ", " " " $t \to t/b$

$$= \lim_{z\to 0} -(a^{-z} - b^{-z}) \int_0^\infty dt\, t^{z-1} e^{-t}$$

$$= \ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right) \quad ✓$$

$(a>0,\ b>0)$

If we had many a's and b's (though equal number of each), the identity becomes:

$$-\int_0^\infty dt\, \frac{1}{t}\left(e^{-ta_1} + \ldots + e^{-ta_n} - e^{-tb_1} - \ldots - e^{-tb_n}\right) = \ln\left(\frac{a_1 a_2 \ldots a_n}{b_1 b_2 \ldots b_n}\right)$$

Assemble all the a's and b's into diagonal matrices:

$$\mathbb{A} = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \quad \& \quad \mathbb{B} = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}$$

Then our identity becomes:

$$-\int_0^\infty dt\, \frac{1}{t}\left(\mathrm{Tr}\left[e^{-t\mathbb{A}}\right] - \mathrm{Tr}\left[e^{-t\mathbb{B}}\right]\right) = \ln\left(\frac{\det \mathbb{A}}{\det \mathbb{B}}\right)$$

or,

$$-\int_0^\infty \frac{dt}{t}\,\mathrm{Tr}\left[e^{-tA} - e^{-tB}\right] = \ln\left(\frac{\det A}{\det B}\right)$$

And since Tr and det are basis invariant, this result applies to any diagonalizable matrix / linear operators, $O(\phi_c)$ & $O(0)$.

$$\boxed{-\int_0^\infty dt\,\frac{1}{t}\,\mathrm{Tr}\left[e^{-t\hat{O}(\phi_c)} - e^{-t\hat{O}(0)}\right] = \ln\left(\frac{\det O(\phi)}{\det O(0)}\right)}$$

Important to pull out trace — causes divergences in QFT.

Schwinger's Proper-time representation of Functional determinants

Notice $e^{-t\hat{O}(\phi_c)}$ and $e^{-t\hat{O}(0)}$ are Heat time-evolution operators

!!

$$\mathrm{Tr}\left[e^{-t\hat{O}}\right] = \underbrace{\sum_{\text{states},\,n} \langle n|\, e^{-t\hat{O}} \,|n\rangle}_{\text{Heat kernel trace}} \equiv \underbrace{\int_{PBC} \mathcal{D}x\; e^{-S_0[x]}}_{\text{Closed-time path representation}} = \int d^dx\, K(x,x;t)$$

since Tr will be taken, it is sufficient to obtain the kernel, $\langle y|e^{-t\hat{O}}|x\rangle$, instead of the full-blown evolution operator, $e^{-t\hat{O}}$. Just set $y=x$, and perform integral over x in the end.

Note: The heat kernel has mass dimension equal the number of space-time dimensions. The heat kernel trace is unitless (because of the d^dx integration):

$e^{-t\hat{O}(\phi)}$ = unitless

$K(x,y;t) \equiv \langle y|e^{-t\hat{O}(\phi)}|x\rangle = [\text{mass}]^d$

$\mathrm{Tr}(x,x;t)$ = unitless

$\hat{O}(\phi) = [\text{mass}]^2$

$t = [\text{mass}]^{-2}$