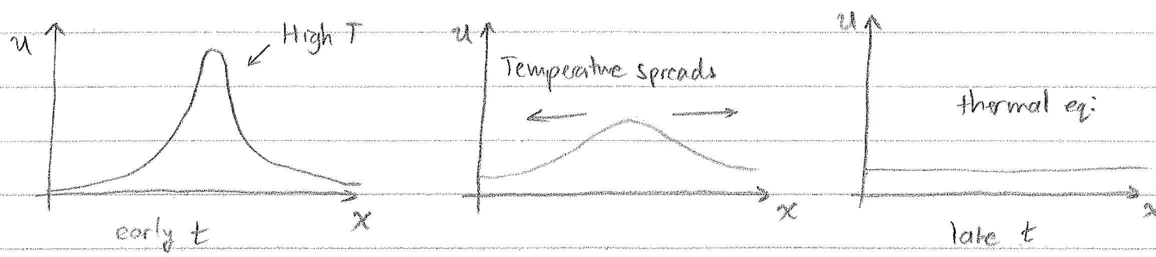
The Heat Equation

$$-\frac{d}{dt}\, u(x,t) = -k \frac{d^2}{dx^2}\, u(x,t)$$

(Minus sign included to ease comparison with QM.)

Notice: $t \to it$ gives the Schrödinger equation for FREE particle if $k = \frac{\hbar}{2m}$

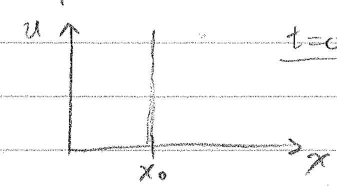
Physically, describes the time (s) evolution of the flow of heat:



Since this is a linear differential equation, as in QM, one can define a Green's function — the solution to the equation for an "impulse" initial condition:

$$u_{x_0}(x, 0) = \delta(x - x_0)$$

Domain: $-\infty < x < +\infty$



The solution — the (time-dependent) Green's function — for this problem is

$$u_{x_0}(x,t) = \frac{1}{\sqrt{4\pi k t}}\, e^{-\frac{(x-x_0)^2}{4kt}} \equiv K(x_0, x, t) \qquad \text{THE HEAT KERNEL}$$

Exercise: Check that this is a solution by plugging back into the Heat equation

c.f. In QM, the Green's function (propagator) for a free particle is (computed using the path integral)

$$\Psi(x_0, x, t) = \sqrt{\frac{m}{2\pi i \hbar t}}\, e^{\frac{im}{2\hbar}\frac{(x-x_0)^2}{t}} \qquad \text{QM KERNEL (for free particle)}$$

can be written: $= \langle x | \hat{U}(t) | x_0 \rangle, \quad \hat{U}(t) = e^{-i\hat{H}t/\hbar}$

Time evolution operator satisfies: $i\hbar \frac{\partial}{\partial t} \hat{U}(t) = \hat{H} U(t)$

In the same sense, one may write a set of states, $\{|x\rangle\}$ upon which the differential operator $-k\,\partial^2/\partial x^2$ acts and write down:

$$K(x_0, x; t) = \langle x|\,\hat{U}(t)\,|x_0\rangle, \qquad \hat{U}(t) = e^{-t\left(-k\frac{d^2}{dx^2}\right)} = e^{-t\hat{O}}$$

Satisfies: $-\dfrac{\partial \hat{U}(t)}{\partial t} = \hat{O}\,\hat{U}(t)$

(Show this)

Can genealize the "free operator" on the RHS of the Heat Equation to include a potential...

$$\hat{O} = -k\frac{d^2}{dx^2} + V(x)$$

...in many dimensions, d:

$$\hat{O} = -k\nabla^2 + V(\vec{x}).$$

For $V(\vec{x}) = 0$, the kernel is:

$$K(\vec{x}_0, \vec{x}; t) = \frac{1}{(4\pi k t)^{d/2}}\, e^{-\frac{(\vec{x}-\vec{x}_0)^2}{4kt}}$$