Gelfand–Yaglom formalism — 1D

Idea: Find a function $\mathcal{F}(\lambda)$ that vanishes at eigenvalues of differential operator $\hat{\mathcal{O}}$. Then one can evaluate function numerically, and relate result to determinant through its zeta function $\zeta'(s=0) = -\ln\det\hat{\mathcal{O}}$.

Analytic function $\mathcal{F}(\lambda_n) = 0$ for all $\{\lambda_n\} \equiv$ eigenvalues of $\hat{\mathcal{O}}$.

(assume well-behaved as $\lambda \to \infty$)

has property: $\dfrac{d}{d\lambda}\ln\mathcal{F}(\lambda) \equiv \dfrac{\mathcal{F}'(\lambda)}{\mathcal{F}(\lambda)}$ has simple poles at λ_n of residue 1

— expand around $\lambda \simeq \lambda_n$

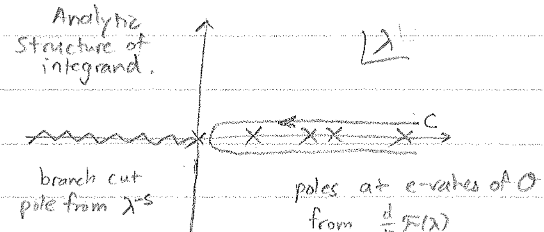
$$\frac{d}{d\lambda}\ln\mathcal{F}(\lambda) = \frac{\mathcal{F}'(\lambda_n)}{\mathcal{F}(\lambda_n) + (\lambda_n - \lambda)\mathcal{F}'(\lambda_n) + \dots} = \frac{1}{\lambda_n - \lambda} + \dots$$

then the zeta function of $\hat{\mathcal{O}}$ is:

$$\zeta(s) = \frac{1}{2\pi i}\int_C d\lambda\, \lambda^{-s}\frac{d}{d\lambda}\ln\mathcal{F}(\lambda)$$

Gelfand–Yaglom representation of $\zeta(s)$

Analytic structure of integrand.



Exercise: Check this — contour encloses all poles:

$$= \frac{1}{2\pi i}\, 2\pi i \sum_n \lambda_n^{-s} = \sum_n \frac{1}{\lambda_n^{-s}}$$

Deform contour around branch cut: The two halves acquire a phase from λ^{-s} (see back)

$$\zeta(s) = \frac{1}{2\pi i}\left[e^{-i\pi s}\int_{-\infty}^{0} d\lambda\,\lambda^{-s}\frac{d}{d\lambda}\mathcal{F}(\lambda) + e^{i\pi s}\int_{0}^{-\infty} d\lambda\,\lambda^{-s}\frac{d}{d\lambda}\mathcal{F}(\lambda)\right]$$

Flip limits $\Rightarrow$ (-1)

$$= \frac{2i\sin(\pi s)}{2\pi i}\int_{-\infty}^{0} d\lambda\,\lambda^{-s}\frac{d}{d\lambda}\ln\mathcal{F}(\lambda)$$

differentiate

$$\zeta'(s) = \cos(\pi s)\int_{-\infty}^{0} d\lambda\,\lambda^{-s}\frac{d}{d\lambda}\ln\mathcal{F}(\lambda) + \frac{\sin\pi s}{\pi}\int_{-\infty}^{0} d\lambda\,(-\lambda^{-s})\ln\lambda\,\frac{d}{d\lambda}\mathcal{F}(\lambda)$$

evaluate at $s=0$ $\Rightarrow$ 2^{nd} term vanishes.

$$\zeta'(0) = \int_{-\infty}^{0} d\lambda \; \frac{d}{d\lambda} \ln \mathcal{F}(\lambda)$$

$$= -\ln \mathcal{F}(0) + \ln \mathcal{F}(-\infty) = -\ln \frac{\mathcal{F}(0)}{\mathcal{F}(-\infty)}$$

Typically independent of details of fluctuation potential — drops out

$$\Rightarrow \boxed{\ln \det \mathcal{O} = -\zeta'(0) = \ln \frac{\mathcal{F}(0)}{\mathcal{F}(-\infty)}}$$

Problem reduced to obtaining $\mathcal{F}(\lambda)$ – usually much easier (especially to obtain numerically). Best to explain with help of example.

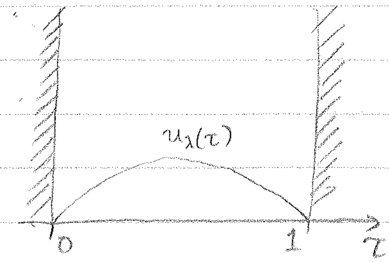
Example: Suppose we want to compute the determinant of $\mathcal{O} = \frac{-\partial^2}{\partial\tau^2} + V(\tau)$ defined in a box: $0 \le \tau \le 1$, with Dirichlet boundary conditions

BOUNDARY-VALUED E-VALUE PROBLEM

$$\left(\frac{\partial^2}{\partial\tau^2} + V(\tau)\right) u_n(\tau) = \lambda_n u_n(\tau) \qquad [1]$$

$$u_n(0) = u_n(1) = 0 \qquad [2]$$

$\Rightarrow$ solutions are quantized.

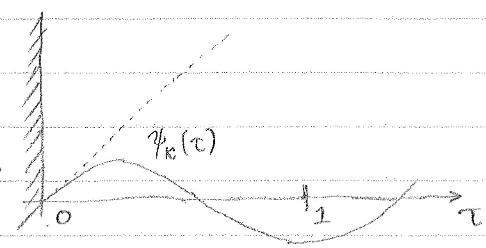


Then consider an associated INITIAL-VALUED E-VALUE PROBLEM

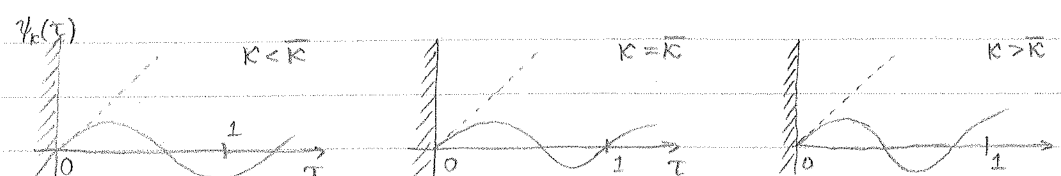
$$\left(\frac{\partial^2}{\partial\tau^2} + V(\tau)\right) \psi_\kappa(\tau) = \kappa\, \psi_\kappa(\tau)$$

with $\psi_\kappa(0) = 0$, $\psi_\kappa'(0) = 1$ ← largely arbitrary: related to arbitrariness of overall normalization.

$\Rightarrow$ solutions are continuous.



For general κ, $\psi_\kappa(1) \neq 0$ (will generally miss it, as shown). However, as we dial κ, $\psi_\kappa(\tau)$ will change, until a special value of $\kappa = \bar{\kappa}$ is reached such that $\psi_{\bar{\kappa}}(1) = 0$. At this point, $\psi_{\bar{\kappa}}(\tau)$ satisfies the boundary conditions of the original BOUNDARY-VALUED PROBLEM, [eqn 2]. Thus, $\bar{\kappa}$ is one of the eigenvalues that goes into the computation of the determinant, i.e. $\bar{\kappa} = \lambda_n$.



also a solution to [1], [2].

Therefore, take $\mathcal{F}(\kappa) \equiv \psi_\kappa(1)$. This is our function which vanishes at eigenvalues of the original BOUNDARY-VALUED PROBLEM.

Then, taking ratio of functional determinants $\mathcal{O} = \frac{-\partial^2}{\partial\tau^2} + V(\tau)$, $\mathcal{O}_{free} = \frac{-\partial^2}{\partial\tau^2}$
(both with same B.C.)

$$\ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{free}}\right) = \left(-\zeta'(0)\right) - \left(-\zeta'_{free}(0)\right)$$

$$= \ln\left(\frac{\mathcal{F}(0)}{\mathcal{F}(-\infty)}\right) - \ln\left(\frac{\mathcal{F}_{free}(0)}{\mathcal{F}_{free}(-\infty)}\right)$$

$$= \ln\left(\frac{\psi_0(1)}{\psi_{-\infty}(1)}\right) - \ln\left(\frac{\psi_0^{free}(1)}{\psi_{-\infty}^{free}(1)}\right)$$

At large e-values, e-functions look like free solutions $\psi_{-\infty}(1) = \psi_{-\infty}^{free}(1)$
(for well behaved potentials)

$$\Rightarrow \boxed{\ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{free}}\right) = \ln\left(\frac{\psi_0(1)}{\psi_0^{free}(1)}\right)}$$

In words: solve the INITIAL-VALUED EIGENVALUE PROBLEM (numerically) for the zero e-value.

That is $\left(\frac{-\partial^2}{\partial\tau^2} + V(\tau)\right)\psi_0(\tau) = 0$ AND $\left(\frac{-\partial^2}{\partial\tau^2}\right)\psi_0^{free}(\tau) = 0$

B.C.: $\psi_0(0) = 0$, $\psi_0'(0) = 1$ $\qquad \psi_0^{free}(0) = 0$ $\psi_0'^{free}(0) = 1$

Then evaluate at $\tau = 1$.

We evaluate at $\tau=1$ because right-hand boundary of box is at $\tau = 1$.

For an infinitely large box, with no right-hand boundary, evaluate at $\tau \to \infty$.