Fermions in a kink soliton

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \bar{\psi}(i\not{\partial})\psi - g\phi\bar{\psi}\psi - V(\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0 \quad\Longrightarrow\quad \boxed{i\not{\partial}\psi - g\phi\psi = 0}$$

In 1+1 Dimensions

$$i\left(\gamma^0\frac{\partial}{\partial t} - \gamma^1\frac{\partial}{\partial x}\right)\psi - g\phi\psi = 0$$

Multiply by γ^0, and isolate $\frac{\partial}{\partial t}$, obtaining time-dep Schrödinger eqn: (Dirac)

$$i\frac{\partial}{\partial t}\psi = i\gamma^0\gamma^1\frac{\partial}{\partial x}\psi + g\phi\gamma^0\psi = \hat{H}_D\psi.$$

Representation: $\gamma^0 = \sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\gamma^1 = i\sigma^2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} =$

$$\gamma^0\gamma^1 = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = -\sigma^3$$

Seek solutions of the form: $\psi(t,x) = e^{-iEt}\psi(x)$ [separation of var.]

$$i(-iE)e^{-iEt}\psi(x) = i\gamma^0\gamma^1 e^{-iEt}\frac{\partial\psi}{\partial x} + g\phi\gamma^0 e^{-iEt}\psi$$

$$i\gamma^0\gamma^1\frac{\partial\psi}{\partial x} + g\underset{\substack{\uparrow\\ \text{Classical Background}\\ \text{scalar field.}}}{\phi}\gamma^0\psi = E\psi$$

Time-independent Schrödinger equation

In the presence of a Classical Background scalar field,

$$\phi_K(t,x) = v\tanh\left(v\sqrt{\frac{\lambda}{2}}(x-x_0)\right) \quad \text{[kink]}$$

with the asymtotics, $\phi_K(x=-\infty) = -v$, $\phi_K(x=+\infty) = +v$, there is a fermion zero mode.

For the zero-mode, $E=0$, and the Schrödinger equation becomes:

$$i\gamma^0\gamma^1 \frac{\partial \psi}{\partial x} + g\phi\gamma^0\psi = 0$$

$$\begin{pmatrix} -i\frac{\partial}{\partial x} & g\phi \\ g\phi & i\frac{\partial}{\partial x} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

$$\begin{cases} -i\frac{\partial}{\partial x}\psi_1 + g\phi\,\psi_2 = 0 & \longleftarrow \text{multiply by } i \\ i\frac{\partial}{\partial x}\psi_2 + g\phi\,\psi_1 = 0 \end{cases}$$

Subtract and add the equations for new equations in terms of $\psi_2 \pm i\psi_2$:

$$\frac{\partial}{\partial x}(\psi_1 + i\psi_2) + g\phi(\psi_1 + i\psi_2) = 0$$

$$\frac{\partial}{\partial x}(\psi_1 - i\psi_2) - g\phi(\psi_1 - i\psi_2) = 0$$

Solve for $\psi_\pm = \psi_1 \pm i\psi_2$:

$$\psi_\pm(x) = \mathcal{N}_\pm\, e^{\pm\int_0^x dx'\, g\phi(x')}$$

Note asymptotics: as $x\to\infty$ $\int_0^x dx'\, g\,\phi(x') \sim gv\int_0^\infty dx' \sim +\infty$

as $x\to-\infty$ $\quad$ " $\sim -gv\int_0^{-\infty} dx' \sim +\infty$

So, normalizable solutions are given by <u>minus sign</u> in the exponent:

$$\psi_-(x) = \mathcal{N}\, e^{-\int_0^x dx'\, g\phi(x')}$$

$\psi_+(x) = 0$ (not normalizable)

In terms of original ψ_1 and ψ_2, we have:

$$\psi_1 = \frac{1}{2}(\psi_+ + \psi_-)$$

$$\psi_2 = \frac{-i}{2}(\psi_+ - \psi_-)$$

$$\Rightarrow \boxed{\psi = \begin{pmatrix} 1 \\ i \end{pmatrix} \mathcal{N} e^{-\int_0^x dx' \, g\phi(x')}}$$

For the kink solution, $\phi(x') = v \tanh\left(v\sqrt{\frac{\lambda}{2}}\, x\right)$ $\longleftarrow$ centered at zero.

Then $\displaystyle\int_0^x dx' \, g \, \phi(x') = \frac{2g}{\sqrt{\lambda}} \ln \cosh\left(v \frac{\sqrt{\lambda}}{2} x\right)$