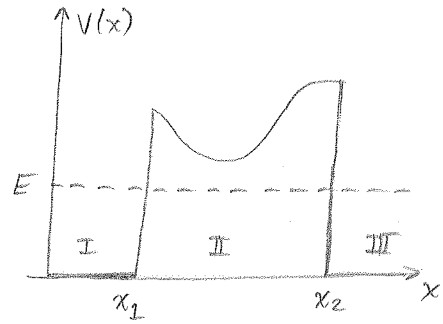
Tunneling through a Potential Barrier

Potential:



- Particle initially in region I; with energy E.
- Particle tunnels through barrier in II, into region III.

Tunneling probability $\sim$ Transmission coef.

$$\psi_I(x) = \psi_{inc}(x) + \psi_{ref}(x) = A e^{ip_0 x/\hbar} + B e^{-ip_0 x/\hbar}$$

$$\psi_{III}(x) = \psi_{trans}(x) = E e^{ip_0 x/\hbar}$$

Free plane-waves in regions I & III.

Use WKB approx. to get wave function, $\psi_{II}(x)$, in region II.

In classically forbidden region, the solution is

$$\psi_{II}(x) = \frac{C}{|p(x)|^{1/2}} \exp\left[-\frac{1}{\hbar}\int dx\, |p(x)|\right] + \underbrace{\frac{D}{|p(x)|^{1/2}} \exp\left[+\frac{1}{\hbar}\int dx\, |p(x)|\right]}_{\text{This increases exponentially for large barrier width -- unphysical} \Rightarrow \text{drop. } D=0}$$

Use continuity equations:

$$\psi_I(x_1) = \psi_{II}(x_1), \qquad \psi_I'(x_1) = \psi_{II}'(x_2) \qquad (1)$$

$$\psi_{II}(x_2) = \psi_{III}(x_2), \qquad \psi_{II}'(x_2) = \psi_{III}'(x_3) \qquad (2)$$

to get B, C, E in terms of A.

$$(1): \quad A e^{ip_0 x_1/\hbar} + B e^{-ip_0 x_1/\hbar} = \frac{C}{|p(x_1)|^{1/2}} \overbrace{\exp\left[-\frac{1}{\hbar}\int_{x_1}^{x_1} dx\, |p(x)|\right]}^{1}$$

and
$$\frac{ip_0}{\hbar}\left(A e^{ip_0 x_1/\hbar} - B e^{-ip_0 x_1/\hbar}\right) = -\frac{p(x_1)}{\hbar |p(x_1)|^{1/2}}\, C \times (1).$$

$$(2): \quad \frac{C}{|p(x_2)|^{1/2}} \exp\left[-\frac{1}{\hbar}\int_{x_1}^{x_2} dx\, |p(x)|\right] = E e^{ip_0 x_2/\hbar}$$

$$-\frac{p(x_2)}{\hbar |p(x_2)|^{1/2}}\, C \exp\left[-\frac{1}{\hbar}\int_{x_1}^{x_2} dx\, |p(x)|\right] = \frac{ip_0}{\hbar} E e^{ip_0 x_2/\hbar}$$

Add two equations of (2): B terms cancel in LHS (after moving $\frac{ip_0}{\hbar}$ to RHS),

$$2 A e^{ip_0 x/\hbar} = \frac{C}{|p(x_1)|^{1/2}} + \frac{i p(x_1)}{p_0} \frac{1}{|p(x_1)|^{1/2}} C = \left(\frac{1}{|p(x_1)|^{1/2}} + \frac{i |p(x_1)|^{1/2}}{p_0} \right) C$$

$$\Rightarrow \quad C = \frac{2 A e^{ip_0 x_1/\hbar}}{\left(\frac{1}{|p(x_1)|^{1/2}} + \frac{i|p(x_1)|^{1/2}}{p_0} \right)} = \frac{2 |p(x_1)|^{1/2} A e^{ip_0 x_1/\hbar}}{1 + i p(x_1)/p_0}$$

Now plug into first equation of (2) to get:

$$E = \frac{C e^{-ip_0 x_2/\hbar}}{|p(x_2)|^{1/2}} \exp\left[-\frac{1}{\hbar} \int_{x_1}^{x_2} dx \, |p(x)| \right]$$

$$= 2 \frac{|p(x_1)|^{1/2}}{|p(x_2)|^{1/2}} \frac{A e^{ip_0 (x_1 - x_2)/\hbar}}{1 + i p(x_1)/p_0} \exp\left[-\frac{1}{\hbar} \int_{x_1}^{x_2} dx \, |p(x)| \right]$$

So, the probability of tunneling (or coefficient of transmission) is

$$(\text{Prob}) = \frac{|E|^2}{|A|^2} = \underbrace{4 \frac{p(x_1)}{p(x_2)} \frac{1}{(1 + i p(x_1)/p_0)^2}}_{\text{some \# of order} \sim 1} \underbrace{\exp\left[-\frac{2}{\hbar} \int_{x_1}^{x_2} dx \, |p(x)| \right]}_{\text{Important exponential suppression}} =$$

$$T \sim \exp\left[\frac{-2}{\hbar} \int_{x_1}^{x_2} dx \, \sqrt{2m(V(x) - E)} \right]$$

"semiclassical tunneling exponent"