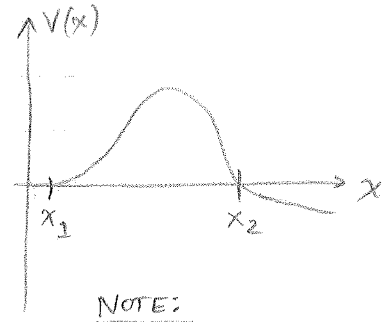
Single-Variable Tunneling (Euclideanized Action)

The semiclassical tunneling exponent can be calculated by evaluating the action in imaginary time – (more rigorously done using path integrals)



Action of one-particle (non-relativistic) system.

$$S = \int_{t_i}^{t_f} dt \left(\frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - V(x) \right)$$

Trick: go to imaginary time; ch. var. $t \to (-i\tau)$

$$S = \int_{-i\tau_i}^{-i\tau_f} d(-i\tau) \left(\frac{1}{2} m \left(\frac{dx}{d(-i\tau)} \right)^2 - V(x) \right)$$

$$= \int_{-i\tau_i}^{-i\tau_f} d(+i\tau) \left(+ \frac{1}{2} m \left(\frac{dx}{d\tau} \right)^2 + V(x) \right)$$

$$= i \int_{-i\tau_i}^{-i\tau_f} d\tau \left[\frac{1}{2} m \left(\frac{dx}{d\tau} \right)^2 + V(x) \right]$$

NOTE:

As presented, reason for this trick is to match WKB. Its origins stem from the path integral method.

And now change integration limits to $+\tau_i \to +\tau_f$ (Wick Rotation) giving the Euclidean Action

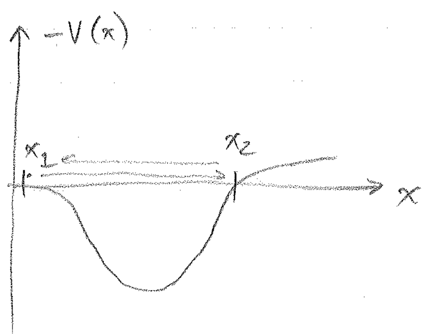
$$i S_E = i \int_{\tau_i}^{\tau_f} \left[\frac{1}{2} m \left(\frac{dx}{d\tau} \right)^2 + V(x) \right]$$

Now extremize this action (obtain E-L eqn. of mot.)

$$\frac{\partial L}{\partial x} - \frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{x}} \right) = 0 \quad , \qquad L = \frac{1}{2} m \left(\frac{dx}{d\tau} \right)^2 + V(x)$$

$$\Rightarrow \boxed{m \frac{d^2 x}{d\tau^2} = \frac{dV}{dx}}$$

This equation of motion formally coincides with Newton's eqn, but with the sign of the potential reversed.



To calculate tunneling probability (in semiclassical approx.), follow two steps:

(1) – solve eqn. of mot. with particle at x_1 initially at rest, rolling into the well at right, bouncing at x_2, and returing to the initial location, x_1. This solution is the "bounce" solution, $x_B(\tau)$.

First integral of the E-L eqn. of motion gives

$$\int m \frac{d^2x}{d\tau^2}\, dx = \int dV \quad \Longrightarrow \quad \underbrace{\frac{1}{2} m \left(\frac{dx}{d\tau}\right)^2}_{KE} \underbrace{- V(x)}_{PE} = E_{eucl.}$$

(see facing page)

$\uparrow$ const. of integration.

For a bounce solution, KE at x_1 vanishes, and $V(x_1) = 0 \Longrightarrow E_{eucl} = 0$ (at all times since E_{eucl} is conserved).

Hence, the bounce solution satisfies

$$\underbrace{\frac{1}{2} m \left(\frac{dx_B}{d\tau}\right)^2 = V(x_B)}_{T \,=\, V \quad \text{(Virial thm)}} \quad \Longrightarrow \quad \frac{dx_B}{d\tau} = \sqrt{\frac{2V(x_B)}{m}} \quad \text{or} \quad d\tau = \sqrt{\frac{m}{2V}}\, dx_B .$$

(2) Then, the tunneling semiclassical exponent is $T \sim e^{-S_E[x_B(\tau)]}$.

$$S_E[x_B] = \int_{\tau_i = -\infty}^{\tau_f = +\infty} \left[\frac{1}{2} m \left(\frac{dx_B}{d\tau}\right)^2 + V(x_B(\tau))\right] d\tau \quad = \left(\int_{-\infty}^{+\infty} m \left(\frac{dx_B}{d\tau}\right)^2 d\tau \right)$$

$$= \int_{-\infty}^{+\infty} d\tau \; 2V(x_B(\tau)),$$

since action is even in x, and solution even in τ,

$$= 2\int_{-\infty}^{0} d\tau \; 2V(x_B(\tau))$$

subs. $d\tau = \sqrt{\frac{m}{2V}}\, dx_B$

$$= 2\int_{x_1}^{x_2} dx_B \sqrt{2mV(x_B)},$$

in agreement with WKB, and matching wavefunctions, for $E = 0$.