Lehmann Ellipse – For Yukawa potentials. $U(r) = \frac{g e^{-\mu r}}{r}$

Region of analyticity in q^2 plane.

Partial wave series: $f(E, q^2) = \sum_{l=0}^{\infty} (2l+1) f_l(E) P_l(\cos\theta)$ (*)

Continue these into complex l- and complex $\cos\theta$-plane.

Bounds for large l:

	physical l for all physical θ	large l complex θ
L. Poly:	$\lvert P_l(\cos\theta)\rvert \le 1$	$\lvert P_l(\cos\theta)\rvert \le \tau(\theta) \frac{e^{\lvert \mathrm{Im}\,\theta\rvert l}}{\sqrt{l}}$ (grows exponentially)
Partial Amplitude:	Generally, $\lvert f_l(E)\rvert < \frac{(\text{const})\, k^{\epsilon}}{l^{2+\epsilon}}$	For Yukawa potential (see DeAlfaro, Regge 1965, § 8.4) $\lvert f_l(E)\rvert < \sigma(E) \frac{e^{-\gamma(E) l}}{\sqrt{l}}$, $\cosh\gamma = 1 + \frac{\mu^2}{2k^2}$ (decays exponentially)

For physical θ, partial wave series (*) is convergent

For complex θ (complex q^2), need exponentially decaying partial-wave amplitudes for convergent partial-wave expansion.

Region of analyticity for Yukawa potentials

Bound on partial wave series (*):

$$|f(E,q^2)| < (\text{const})\, \tau(\theta)\, \sigma(E) \sum_{l=0}^{\infty} \frac{e^{(|\mathrm{Im}\,\theta| - \gamma) l}}{l}, \qquad \cosh\gamma = 1 + \frac{\mu^2}{2k^2}$$

$\Rightarrow$ $f(E,q^2)$ analytic where

$$|\mathrm{Im}\,\theta| < \gamma$$ (so that sum converges as $l \to \infty$)

Find region of analyticity in $z = \cos\theta$ - plane.

Let $\theta = \theta_1 + i\theta_2$

$$\Rightarrow z(\theta_1, \theta_2) = \cos(\theta_1 + i\theta_2) \qquad [\,\mathrm{Im}\,\theta = \theta_2\,]$$

$$= \frac{1}{2}\left[e^{i(\theta_1 + i\theta_2)} + e^{-i(\theta_1 + i\theta_2)} \right]$$

$$z(\theta_1,\theta_2) = \frac{1}{2}\left[e^{i\theta_1} e^{-\theta_2} + e^{-i\theta_1} e^{\theta_2}\right]$$

$$= \frac{1}{2}\left[(\cos\theta_1 + i\sin\theta_1)\, e^{-\theta_2} + (\cos\theta_1 - i\sin\theta_1)\, e^{\theta_2}\right]$$

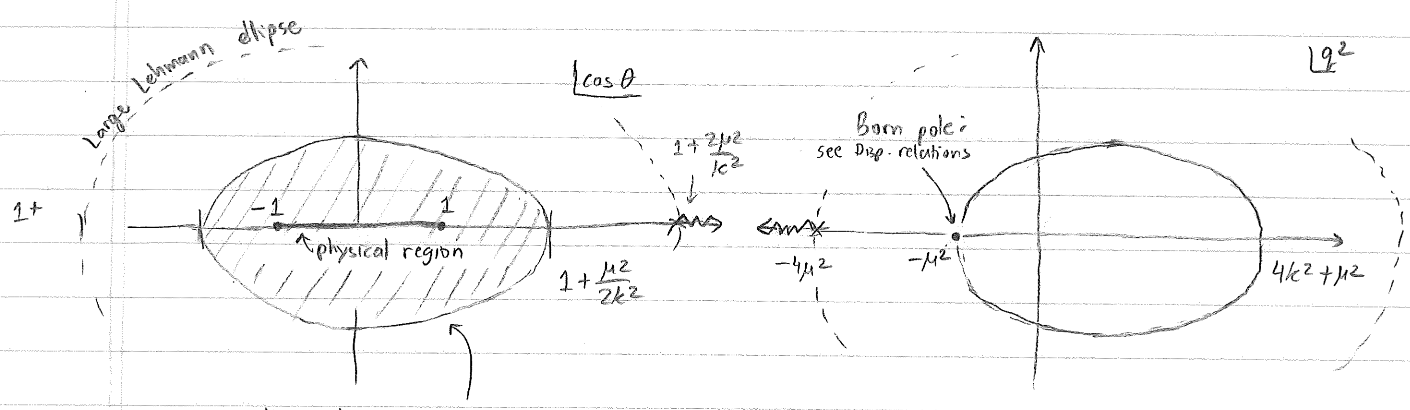
$$= \frac{1}{2}\left[\cos\theta_1 \left(e^{-\theta_2} + e^{\theta_2}\right) + i\sin\theta_1 \left(e^{-\theta_2} - e^{\theta_2}\right)\right]$$

$$= \cos\theta_1 \cosh\theta_2 - i\sin\theta_1 \sinh\theta_2$$

Look at boundary: $\mathrm{Im}\,\theta \equiv \theta_2 = \gamma \Rightarrow \begin{cases} \cosh\theta_2 = 1 + \frac{\mu^2}{2k^2} \\ \sinh\theta_2 = \sqrt{\frac{\mu^2}{2k^2}\left(2+\frac{\mu^2}{2k^2}\right)} \end{cases}$

$$z(\theta_1, \theta_2^{\text{Bound}}) = \left(1+\frac{\mu^2}{2k^2}\right)\cos\theta_1 - i\sqrt{\frac{\mu^2}{2k^2}\left(2+\frac{\mu^2}{2k^2}\right)}\ \sin\theta_1$$

$\rightarrow$ parametric plot: $0 < \theta_1 < 2\pi$ to obtain boundary of analytic region.



Small Lehmann ellipse
(REGION OF ANALYTICITY)

$$q^2 = 2k^2(1-\cos\theta)$$

$$q^2_{\text{Left}} = 2k^2\left(\cancel{1} - \left(\cancel{1}+\frac{\mu^2}{2k^2}\right)\right) = \mu^2$$

$$q^2_{\text{Right}} = 2k^2\left(1 + \left(1+\frac{\mu^2}{2k^2}\right)\right) = 4k^2+\mu^2$$

Region of analyticity reflects the position of Born pole: $\frac{-2gm}{\mu^2+q^2} = f_{\text{Born}}$

By studying analyticity of $f(E,q^2) - f_{\text{Born}}$, can obtain larger region of analyticity:

LARGE LEHMANN ELLIPSE: with $\mu^2 \rightarrow 4\mu^2$.