## Relation to resonance widths

$z = \cos\theta$

Use Sommerfeld-Watson representation to obtain partial wave amplitude:

$$f_\ell(E) = \frac{1}{2}\int_{-1}^{1} dz \; f(E, q^2(z))\, P_\ell(z)$$

$$= \frac{1}{2}\int_{-1}^{1} dz \left[ \frac{1}{2i}\int_{-\frac{1}{2}+i\infty}^{\frac{1}{2}-i\infty} d\ell' \, \frac{(2\ell'+1) f_{\ell'}(E) P_{\ell'}(-z)}{\sin(\pi\ell')} + \sum_n \frac{\beta_n(E) P_{\alpha_n(E)}(-z)}{\sin \pi\alpha_n(E)} \right] P_\ell(z)$$

Identity: $\frac{1}{2}\int_{-1}^{1} dz\, P_{\ell'}(-z) P_\ell(z) = \frac{1}{\pi} \frac{\sin \pi \ell'}{(\ell'-\ell)(\ell'+\ell+1)}$

$$f_\ell(E) = \frac{1}{2\pi i}\int_{-\frac{1}{2}+i\infty}^{\frac{1}{2}-i\infty} d\ell' \, (2\ell'+1) \frac{f_{\ell'}(E)}{(\ell'-\ell)(\ell'+\ell+1)} + \frac{1}{\pi}\sum_n \beta_n(E) \frac{1}{(\alpha_n(E)-\ell)(\alpha_n(E)+\ell+1)}$$

Below threshold $(E<0)$, $\alpha_\ell(E)$ are real $\Rightarrow$ gives rise to poles on negative real-$E$ axis $\rightarrow$ Bound state.
Above threshold $(E>0)$, expect $\alpha_\ell(E)$ to become complex.

Suppose at some $E = E_0$, $\mathrm{Re}\, \alpha(E_0) = \text{integer} = \ell$,

approx: $\alpha(E) \simeq \alpha(E_0) + (E-E_0)\frac{d\alpha(E_0)}{dE}$

$$= \alpha_R(E_0) + i\alpha_I(E_0) + (E-E_0)\frac{d\alpha_R(E_0)}{dE} + \ldots \quad \text{(linear approx)}$$

Then

$$f_\ell(E)\Big|_{\substack{\text{pole}\\ \text{term}}} = \frac{1}{\pi} \frac{\beta_n(E_0)}{\alpha_n(E_0)+\ell+1} \frac{1}{\cancel{\alpha_R(E_0)} + i\alpha_I(E_0) + (E-E_0)\alpha_R'(E_0) \cancel{-\ell}}$$

$$= \frac{1}{\pi}\frac{\beta_n(E_0)}{\alpha_n(E_0)+\ell+1} \frac{1}{(E-E_0)\alpha_R'(E_0) + i\alpha_I(E_0)}$$

$\longrightarrow$ has Breit-Wigner form, with width

$$\Gamma \propto \frac{\alpha_I(E_0)}{\alpha_R'(E_0)}$$