Mandelstam Representation

Having established analyticity of FULL SCATTERING AMPLITUDE, $f(E,q^2)$ in q^2, can proceed to find double dispersion in E and q^2.

Recall, dispersion relation for NON-FORWARD scattering amplitude, for q^2 Reals.

$$f(E,q^2) = \frac{-2gm}{\mu^2+q^2} - \sum_n \frac{\text{Res } f(E,0) \times P_{l_n}(\cos\theta)}{E_n - E} + \frac{1}{\pi}\int_0^\infty dE' \frac{1}{E-E'} \underbrace{\text{Im} f(E',q^2)}_{\text{For complex } q^2\text{, not quite imaginary part}}$$

For complex q^2, must write:

$$= \frac{-2gm}{\mu^2+q^2} + (\text{Poles}) + \frac{1}{\pi}\int_0^\infty dE' \frac{1}{E-E'} \times \frac{1}{2i}\left[f(E'+i\epsilon, q^2) - f(E'-i\epsilon, q^2)\right]$$

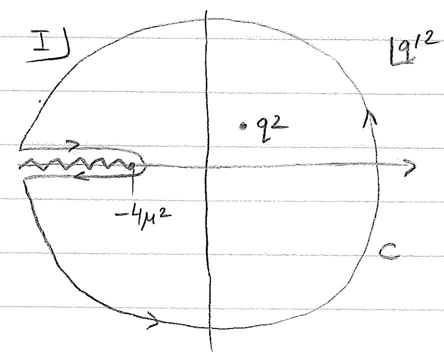
because $f^*((E+i\epsilon)^*, (q^2)^*) \neq f(E-i\epsilon, q^2)$ for complex q^2.

Let $\frac{1}{2i}\left[f(E'+i\epsilon, q^2) - f(E'-i\epsilon, q^2)\right] \equiv I(E', q^2)$, and derive dispersion relation in q^2.

Sommerfeld rep + Lehmann Ellipse $\Rightarrow$ cut in q^2 from $-\infty \to -4\mu^2$

There is a Born pole at $q^2 = -\mu^2$, but cancels in difference.

Start with



$$2\pi i\, I(E', q^2) = \oint_C dq'^2 \frac{I(E', q'^2)}{q'^2 - q^2}$$

$$= \int_{\substack{\text{Great}\\ \text{circle}}} + \int_{-\infty}^{-\mu^2} dq'^2 \frac{I(E', q'^2 + i\epsilon)}{q'^2 - q^2} + \int_{-\mu^2}^{-\infty} dq'^2 \frac{I(E', q'^2 - i\epsilon)}{q'^2 - q^2}$$

Must know $q^2 \to \infty$ behavior of $f(E,q^2)$ (Regge theory - bound states will require subtractions)

Assume $f(E,q^2) \to 0$ for now.

$$\Rightarrow \quad I(E', q^2) = \frac{1}{\pi}\int_{-\infty}^{-4\mu^2} dq'^2 \frac{1}{q'^2 - q^2} \underbrace{\frac{1}{2i}\left[I(E', q'^2 + i\epsilon) - I(E', q'^2 - i\epsilon)\right]}_{\text{Define: } \equiv \rho(E', q'^2)}$$

Double spectral function

(Defined only for $0 \le E' < \infty$ & $-\infty < q'^2 \le -4\mu^2$)

$$f(E,q^2) = \frac{-2gm}{\mu^2+q^2} + (\text{Poles}) + \frac{1}{\pi^2}\int_0^\infty \frac{dE'}{E'-E}\int_{-\infty}^{-4\mu^2} \frac{dq'^2}{q'^2 - q^2}\,\rho(E', q'^2)$$

(provided no subtractions needed)

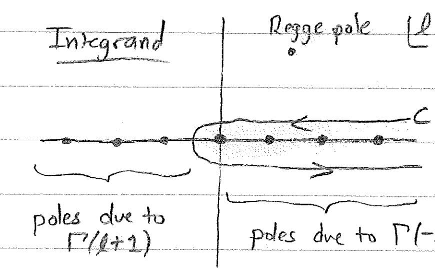
Sommerfeld–Watson Transformation with Γ-functions (aside)

$$f(E,q^2) = \sum_\ell (2\ell+1)\, f_\ell(E)\, P_\ell(\cos\theta)$$

Replace with integral

$$f(E,q^2) = \frac{-1}{2\pi i}\oint_C d\ell\,(2\ell+1)\, f_\ell(E)\, P_\ell(-\cos\theta) \times \overbrace{\Gamma(-\ell)\Gamma(\ell+1)}^{-\pi/\sin\pi\ell}$$

check: $\Gamma(-\ell)$ generates poles at (positive) integer ℓ with residue $\dfrac{-(-1)^\ell}{\Gamma(\ell+1)}$



Cauchy's residue theorem recovers partial wave series:

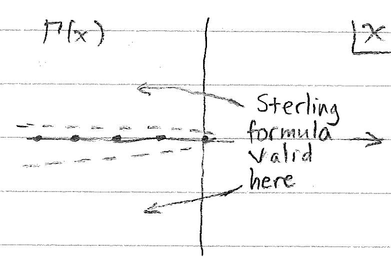
$$f(E,q^2) = \frac{\cancel{-}1}{\cancel{2\pi i}}\,\cancel{2\pi i}\sum_{\ell=0}^{\infty}(2\ell+1)\, f_\ell(E)\, \underbrace{P_\ell(-\cos\theta)\times\cancel{-}(-1)^\ell}_{P_\ell(\cos\theta)}\,\frac{\cancel{\Gamma(\ell+1)}}{\cancel{\Gamma(\ell+1)}}$$

$$= \sum_\ell (2\ell+1)\, f_\ell(E)\, P_\ell(\cos\theta) \quad \checkmark$$

Does the amplitude admit a Sommerfeld–Watson representation?
i.e. does the integrand vanish quickly enough with $|\ell|\to\infty$ to allow deforming the contour?

check using Sterling formula:

$$\Gamma(x) \xrightarrow[|x|\to\infty]{} e^{-x}\, x^{x-1/2}\,(2\pi)^{1/2}$$



write $x=\rho e^{i\theta}$ $\quad -\pi<\theta<\pi \quad (\theta\neq\pi)$

$$\Gamma(x) \longrightarrow (\text{phase})\,\exp\left[\mathrm{Re}(x)(\ln\rho-1)-\tfrac{1}{2}\ln\rho-\theta\,\mathrm{Im}(x)\right]$$

Therefore: $\quad \Gamma(-\ell)\Gamma(\ell+1) \longrightarrow \dfrac{1}{|\ell|}e^{-\pi|\mathrm{Im}(\ell)|}$ (away from real-ℓ axis)

Provided $f_\ell(E) \xrightarrow[|\ell|\to\infty]{} 0$, the amplitude <u>does</u> admit the Sommerfeld–Watson transformation.