The Legendre functions

Representation functions of $SO(3)$ are the spherical harmonics:

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi}$$

Associated Legendre polynomials.

Azimuthal symmetry around z-axis:

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)$$

Legendre function solutions to:

$$\hat{L}^2 P_l(\cos\theta) = l(l+1)\, P_l(\cos\theta)$$

or

$$\frac{d}{dz}\left[(1-z^2)\frac{dP_l}{dz}\right] + l(l+1)P_l = 0$$

Legendre's equation

$z \equiv \cos\theta$

Regular solutions: Legendre functions of the FIRST KIND

- regular at all z for real integer l

For integer l, solutions are Legendre polynomials

$$P_{l=0}(z) = 1$$
$$P_{l=1}(z) = z$$
$$P_{l=2}(z) = \frac{1}{2}(3z^2-1)$$
$$\vdots$$

For non-integer l, solutions expressed as hypergeometric functions

$$P_l(z) = {}_2F_1\left(-l, l+1; 1; \frac{1-z}{2}\right)$$

Irregular solutions: Legendre functions of the SECOND KIND

- singular at $z = \pm 1$ for real integer l.

For integer l,

$$Q_0(z) = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)$$

$$Q_1(z) = \frac{1}{2}z\ln\left(\frac{z+1}{z-1}\right) - 1$$

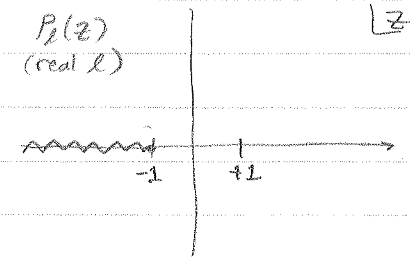
$$Q_2(z) = \frac{1}{2}P_2(z)\ln\left(\frac{z+1}{z-1}\right) - \frac{3}{2}z$$

$$Q_l(z) = \frac{1}{2}P_l(z)\ln\left(\frac{z+1}{z-1}\right) - \sum_{n=1}^{l}\frac{1}{n}P_{n-1}(z)P_{l-n}(z)$$

For non-integer l, solutions are expressed more generally as

$$Q_l(z) = \sqrt{\pi}\,\frac{\Gamma(l+1)}{\Gamma(l+3/2)}\frac{1}{(2z)^{l+1}}\; {}_2F_1\left(\frac{l}{2}+1, \frac{l}{2}+\frac{1}{2}; l+\frac{3}{2}; \frac{1}{z^2}\right)$$

Properties of Legendre functions



- Legendre equation invariant under $\ell \to -\ell-1$, so

$$\Rightarrow \quad P_\ell(z) = P_{-\ell-1}(z)$$

- For $\ell \in \mathbb{R}$, $\quad \operatorname{Im} P_\ell(z) = \begin{cases} -P_\ell(-z)\sin(\pi\ell) & z < -1 \\ 0 & z > 0 \end{cases}$

The two solutions (FIRST & SECOND KIND) are related, for integer ℓ, by

$$Q_\ell(z) = \frac{1}{2}\int_{-1}^{1} \frac{dz'}{z-z'} P_\ell(z'), \qquad \ell = 0, 1, 2, \ldots$$

"dispersion relation for Q"

Integral relations:

$$\int_{-1}^{1} P_\ell(-z) P_\alpha(z)\, dz = \frac{1}{\pi} \frac{2\sin\pi\alpha}{(\alpha-\ell)(\alpha+\ell+1)}, \qquad \ell \in \mathbb{Z}, \quad \alpha \text{ any number.}$$

$$\int_{1}^{\infty} Q_\ell(z) P_{\ell'}(z)\, dz = \frac{1}{(\ell-\ell')(\ell+\ell'+1)}, \qquad \ell, \alpha \text{ any number}$$

$$P_\ell(-z) = \frac{\sin\pi\ell}{\pi} \int_1^\infty \frac{dz'}{z-z'} P_\ell(z')$$

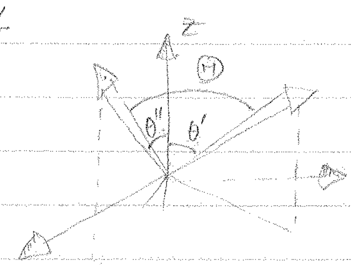
(dispersion relation for P)

Orthogonality relations:

$$\int_{-1}^{1} dz\, P_\ell(z) P_{\ell'}(z) = \frac{2}{2\ell+1}\delta_{\ell\ell'} \qquad \ell \in \mathbb{Z}$$

more generally

$$\int_0^{2\pi} d\phi \int_{-1}^{1} dz\, P_\ell(z') P_{\ell'}(\xi) = \frac{4\pi}{2\ell+1} P_\ell(z'')\delta_{\ell\ell'}$$



$z' = \cos\theta'$
$z'' = \cos\theta''$
$\xi = \cos\Theta$