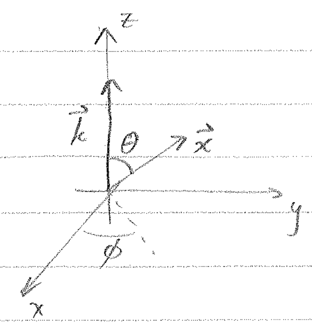
Hiren H. Patel
hnpatel.net

Expansion of (scalar) plane waves into spherical harmonics (Rayleigh expansion)

Result: $$\boxed{e^{ikz} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\, j_\ell(kr)\, P_\ell(\cos\theta)}$$



$\vec{k} = (0, 0, k) = k\hat{z}$

$r = |\vec{x}|$

$\theta \equiv$ angle between $\vec{x}$ & z-axis.

Proof:

[ANSATZ]

Write $$e^{\overbrace{ikz}^{kr\cos\theta}} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} c_{\ell m}(kr)\, Y_\ell^m(\theta,\phi) \qquad (*)$$

Goal: find expansion coefficients $c_{\ell m}(kr)$

To that end, multiply by $Y_{\ell'}^{m'}(\theta,\phi)$ and integrate over θ, ϕ:

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\, e^{ikr\cos\theta}\, Y_{\ell'}^{m'*}(\theta,\phi)$$

$$= \sum_{\ell=0}^{\infty}\sum_{m_\ell=-\ell}^{\ell} c_{\ell m}(kr) \underbrace{\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\, Y_{\ell m}(\theta,\phi)\, Y_{\ell'}^{m'*}(\theta,\phi)}_{\delta_{\ell\ell'}\delta_{mm'} \quad \text{(orthonormality)}}$$

sum over ℓ & m fixing $\ell \to \ell'$, $m \to m'$

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\, e^{ikr\cos\theta}\, Y_{\ell'}^{m'*}(\theta,\phi) = c_{\ell'm'}(kr)$$

recall: $Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi}$

So,

$$\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} \underbrace{\int_0^{2\pi} d\phi\, e^{-im\phi}}_{=\begin{cases}2\pi, & m=0\\ 0, & m\neq 0\end{cases}} \int_{-1}^{1} d(\cos\theta)\, e^{ikr\cos\theta}\, P_{\ell'}^{m'}(\cos\theta) = c_{\ell'm'}(k)$$

rename $\cos\theta \to z$

$\Rightarrow$ $c_{\ell m}(kr)$ is non-zero for $m=0$ only.

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Set $m=0$.

$$\sqrt{\frac{2l+1}{4\pi}\frac{l!}{l!}}\,(2\pi)\,\delta_{m'0}\underbrace{\int_{-1}^{1} dz\, e^{ikrz}\, P_{l'}(z)}_{\text{Poisson integral } = \, 2i^{l'} j_{l'}(kr)} = C_{l'}(kr)$$

$$\underline{\underline{\sqrt{(2l+1)\pi}\cdot 2i^{l}\, j_l(kr) = C_l(kr)}}$$

Therefore, plugging back into $(*)$:

$$e^{ikz} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sqrt{(2l+1)\pi}\; 2i^{l}\,\delta_{m0}\; j_l(kr)\; Y_l^m(\theta,\phi)$$

$$= \sum_{l=0}^{\infty}\sqrt{(2l+1)\pi}\; 2i^{l}\, j_l(kr)\,\underbrace{Y_l^{m=0}(\theta,\phi)}_{\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)}$$

IMPORTANT: only $m=0$ harmonics contribute

$$\boxed{e^{ikz} = \sum_{l=0}^{\infty} i^{l}\,(2l+1)\, j_l(kr)\; P_l(\cos\theta)}$$