Generalization: Gegenbauer expansion

For plane waves in a general direction,
first orient quantization axis $\hat{s}$ (of orb. ang. mom.) along $\vec{k}$.

Same result:

reinterpret angle: Θ = between $\vec{x}$ & $\vec{k}$ ← which is also $\hat{s}$.

$$e^{i\vec{k}\cdot\vec{x}} = \sum_{l=0}^{\infty} \sqrt{4\pi(2l+1)}\, i^l j_l(kr)\, Y_l^{m=0}(\Theta, \phi)$$

$$\left(= \sum_{l=0}^{\infty} i^l (2l+1)\, j_l(kr)\, P_l(\cos\Theta) \right)$$

Then "rotate" quantization axis using addition theorem for $Y_l^m(\theta,\phi)$:

$$\sum_{m=-l}^{l} \underbrace{Y_l^{*m}(\theta',\phi')}_{\text{direction angle for } \vec{k}} \underbrace{Y_l^m(\theta,\phi)}_{\text{direction angle for } \vec{x}} = \sqrt{\frac{2l+1}{4\pi}}\, Y_l^0(\Theta, \phi)$$

with respect to quantization axis $\hat{s}$.

angle between $\vec{x}$ & $\vec{k}$.

$$\Rightarrow\ e^{i\vec{k}\cdot\vec{x}} = \sum_{l=0}^{\infty} \sqrt{4\pi(2l+1)}\, i^l j_l(kr) \sqrt{\frac{4\pi}{2l+1}} \sum_{m=-l}^{l} Y_l^{*m}(\Omega_{\vec{k}})\, Y_l^m(\Omega_{\vec{x}})$$

$$= 4\pi \sum_{l=0}^{\infty} i^l j_l(kr) \sum_{m=-l}^{l} Y_l^{*m}(\Omega_{\vec{k}})\, Y_l^m(\Omega_{\vec{x}})$$

⟹ Can infer generalization to Poisson's integral:

$$\int d\Omega_{\vec{x}}\, Y_l^m(\Omega_{\vec{x}})\, e^{i\vec{k}\cdot\vec{x}} = 4\pi i^l j_l(kr)\, Y_l^m(\Omega_{\vec{k}})$$