Normalization of spherical Bessel functions

Start with

$$\int d^3x \; e^{i\vec{k}\cdot\vec{x}} \; e^{-i\vec{k}'\cdot\vec{x}} = (2\pi)^3 \, \delta^{(3)}(\vec{k}-\vec{k}')$$

insert multipole expansions

$$\text{LHS} = \sum_{l,m}\sum_{l',m'} (4\pi)^2 \, i^{l-l'} \int d^3x \; j_l(kr)\, j_{l'}(k'r) \; Y_l^{m*}(\theta_{\vec{k}})\, Y_l^m(\theta_{\vec{x}})\, Y_{l'}^{m'}(\theta_{\vec{k}'})\, Y_{l'}^{*m'}(\theta_{\vec{x}})$$

$$= \sum_{l,m}\sum_{l',m'} (4\pi)^2 \, i^{l-l'} \; Y_l^{*m}(\theta_{\vec{k}})\, Y_{l'}^{m'}(\theta_{\vec{k}'}) \underbrace{\int d\Omega_x \; Y_l^m(\theta_{\vec{x}})\, Y_{l'}^{*m'}(\theta_{\vec{x}})}_{\delta_{ll'}\delta_{mm'}} \int dr\, r^2 j_l(kr)\, j_{l'}(k'r)$$

$$= \sum_{l,m} (4\pi)^2 \; Y_l^{*m}(\theta_{\vec{k}})\, Y_l^m(\theta_{\vec{k}'}) \int dr\, r^2 j_l(kr)\, j_l(k'r)$$

$$\text{RHS} = (2\pi)^3 \, \delta^{(3)}(\vec{k}-\vec{k}')$$

spherical polars.

$$= (2\pi)^3 \frac{1}{k^2 \sin\theta} \, \delta(k-k')\, \delta(\theta_{\vec{k}}-\theta_{\vec{k}}')\, \delta(\phi_{\vec{k}}-\phi_{\vec{k}}')$$

completeness.

$$= (2\pi)^3 \frac{1}{k^2} \, \delta(k-k') \sum_{l,m} Y_l^{*m}(\theta_{\vec{k}})\, Y_l^m(\theta_{\vec{k}'})$$

Match RHS & LHS term by term (l, m).

$$(4\pi)^2 \int dr\, r^2 j_l(kr)\, j_l(k'r) = (2\pi)^3 \frac{1}{k^2} \, \delta(k-k')$$

$$\therefore \; \boxed{\int_0^\infty dr\, r^2 j_l(kr)\, j_l(k'r) = \frac{\pi}{2} \frac{1}{k^2} \, \delta(k-k')}$$

$\Rightarrow$ Free radial solutions conveniently normalized as (i^l included for conv.)

$r R_{kl}(r) = u_{kl}(r)$

$$R_{kl}(r) = i^l \sqrt{\frac{2k^2}{\pi}} \, j_l(kr)$$

multiply by r

$$u_l(k,r) = i^l \sqrt{\frac{2}{\pi}} \, kr \, j_l(kr) = i^l \sqrt{\frac{2}{\pi}} \, \hat{j}_l(kr)$$

Normalization of (free) spherical waves:

$$\psi_{k\ell m}(\vec{x}) = R_{k\ell}(r)\, Y_\ell^m(\theta,\phi) = i^\ell \sqrt{\frac{2}{\pi}}\, \frac{1}{r}\, u_\ell(k,r)\, Y_\ell^m(\theta,\phi)$$

with $\qquad\qquad\qquad\qquad$ or

$$R_{k\ell}(r) = i^\ell \sqrt{\frac{2k^2}{\pi}}\, j_\ell(kr) \qquad\qquad u_\ell(k,r) = \hat{j}_\ell(kr)$$

Completeness:

$$\int_0^\infty dk \sum_{\ell,m} \psi^*_{k\ell m}(\vec{x})\, \psi_{k\ell m}(\vec{x}') = \delta^{(3)}(\vec{x}-\vec{x}') \qquad \checkmark$$

Orthonormality:

$$\int d^3x \quad \psi^*_{k\ell m}(r,\theta,\phi)\, \psi_{k'\ell' m'}(r,\theta,\phi) = \delta(k-k')\, \delta_{\ell\ell'}\, \delta_{mm'}$$

Normalization of Riccati-Bessel function:

$$\hat{j}_\ell(kr) = (kr)\, j_\ell(kr)$$

$$\int_0^\infty dr\ \hat{j}_\ell(kr)\,\hat{j}_\ell(k'r) = kk'\int_0^\infty dr\ r^2\, j_\ell(kr)\, j_\ell(k'r)$$

$$= kk'\ \frac{\pi}{2}\frac{1}{k^2}\,\delta(k-k')$$

$$= \frac{\pi}{2}\,\delta(k-k')$$

Normalized Riccati-radial function (properly normalized scattering functions)

$$\int_0^\infty dr\ u_\ell(k,r)\, u_\ell(k',r) = \frac{\pi}{2}\,\delta(k-k')$$

Note: Riccati-radial functions are dimensionless.

Full wavefunction:

$$\langle \vec{x} \,|\, E, \ell, m_\ell \rangle = i^\ell \sqrt{\frac{2m}{\pi k}}\ \frac{1}{r}\, u_\ell(k,r)\ Y_\ell^m(\Omega_x)$$

$$\langle \vec{x} \,|\, k, \ell, m_\ell \rangle = \underbrace{i^\ell \sqrt{\frac{2}{\pi}}\ \frac{1}{r}\, u_\ell(k,r)}_{R_\ell(k,r)}\ Y_\ell^{M}(\Omega_{\vec{x}})$$