Confluent Hypergeometric Function ${}_1F_1(\alpha;\beta;z)$

Kummer's equation

$$\left[z\frac{d^2}{dz^2}+(\beta-z)\frac{d}{dz}-\alpha\right]f(z)=0$$

Confluent Hypergeometric series:

$${}_1F_1(\alpha;\beta;z)=1+\frac{\alpha}{\beta}z+\frac{1}{2!}\frac{\alpha(\alpha+1)}{\beta(\beta+1)}z^2+\mathcal{O}(z^3)$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(\alpha+n)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta+n)}z^n=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\alpha^{(n)}}{\beta^{(n)}}z^n$$

Properties:

- Series well-defined provided $\beta\neq -p$, where p = negative integers or zero
- converges in entire complex z plane.
- If $\alpha=-p$ (zero or negative integer) then series terminates, and is a polynomial of degree p.
  (If $\alpha\neq -p$), then ${}_1F_1(\alpha;\beta;z)$ has an essential singularity at $z\to\infty$.
- Satisfies Kummer relation:
  $${}_1F_1(\alpha;\beta;z)=e^z\,{}_1F_1(\beta-\alpha;\beta;z)$$

If β is not integer, the the (regular) solutions to Kummer's equation are:

$$f_A(z)={}_1F_1(\alpha;\beta;z)\quad\text{and}\quad f_B(z)=z^{1-\beta}\,{}_1F_1(\alpha-\beta+1;2-\beta;z)$$

If $\beta=+1$, these two solutions coincide.

If $\beta=\{0,-1,-2,-3,\dots\}$, only $f_B(z)=z^{1-\beta}\,{}_1F_1(\alpha-\beta+1;2-\beta;z)$ is a solution

If $\beta=\{2,3,4,\dots\}$, then only $f_A(z)={}_1F_1(\alpha;\beta;z)$ is a solution.

The regular solutions are given by the hypergeometric series expansion.

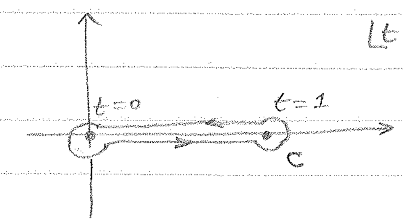
$$ {}_1F_1(\alpha,\beta;z) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \frac{\Gamma(\beta)}{\Gamma(\beta+n)} z^n $$

Messiah, QM I (1961)

Which has the integral representation, for positive integer $\beta = b$.

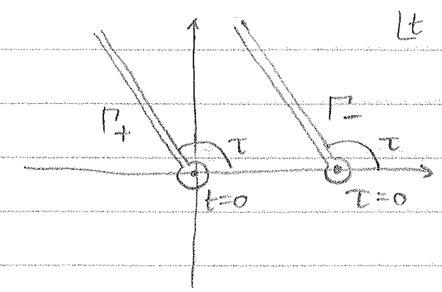
$$ {}_1F_1(\alpha, b; z) = \left(1 - e^{-2\pi i \alpha}\right)^{-1} \frac{\Gamma(b)}{\Gamma(\alpha)\Gamma(b-\alpha)} \oint_C dt\, e^{zt}\, t^{\alpha-1} (1-t)^{b-\alpha-1} $$

POSITIVE INTEGER $b>0$



To obtain the irregular solutions, choose different contours:

$$ W_\pm(\alpha; b; z) = \left(1 - e^{-2\pi i \alpha}\right)^{-1} \frac{\Gamma(b)}{\Gamma(\alpha)\Gamma(b-\alpha)} \oint_{\Gamma_\pm} dt\, e^{zt}\, t^{\alpha-1} (1-t)^{b-\alpha-1} $$



where the τ is argument of location of contour at infinity.

Note: ${}_1F_1(\alpha; b; z) = W_+(\alpha; b; z) + W_-(\alpha; b; z)$

Asympotic (large z) behavior:

$$ W_+(\alpha; b; z) \xrightarrow{|z|\to\infty} \frac{\Gamma(b)}{\Gamma(b-\alpha)} (-z)^{-\alpha} \left[1 + O\left(\tfrac{1}{z}\right)\right] \leftarrow \mathrm{Re}(z) < 0 $$

$$ W_-(\alpha; b; z) \xrightarrow{|z|\to\infty} \frac{\Gamma(b)}{\Gamma(\alpha)} e^z z^{\alpha-b} \left[1 + O\left(\tfrac{1}{z}\right)\right] \leftarrow \mathrm{Re}(z) > 0 $$

is the asymptotic behavior of ${}_1F_1$ if

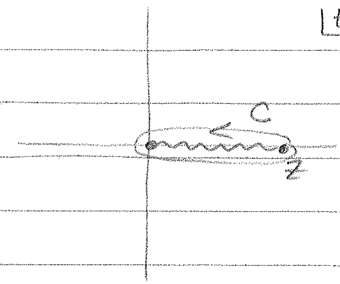
$$ {}_1F_1(\alpha; b; z) \xrightarrow{|z|\to\infty} \left(\frac{\Gamma(b)}{\Gamma(\alpha)} e^z z^{\alpha-b} + \frac{\Gamma(b)}{\Gamma(b-\alpha)} (-z)^{-\alpha} \right) \left[1 + O\left(\tfrac{1}{z}\right)\right] $$

Hecht. QM (2000) ch 42 (eq. 14)

—Other useful integral representations—

If b is an even integer, can write

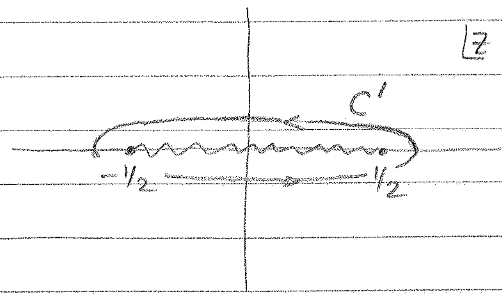
$$ {}_1F_1(\alpha, b, z) = \frac{\Gamma(b)}{2\pi i} \oint_C dt\, e^{t}\, t^{a-b} (t-z)^{-a} $$



ch. var: $t = -z(t' - \frac{1}{2})$

$dt = -z\, dt'$

$$ {}_1F_1(\alpha, b, z) = \frac{\Gamma(b)}{2\pi i} z^{1-b} e^{z/2} \oint_{C'} \frac{dt\, e^{-zt}}{(t-\frac{1}{2})^{b-\alpha} (t+\frac{1}{2})^{\alpha}} $$



More generally,

ch. var: $t = -z(t' - \xi)$ $\xi \in \mathbb{C}$ is an arbitrary parameter.

$dt = -z\, dt'$

$$ {}_1F_1(\alpha, b, z) = \frac{\Gamma(b)}{2\pi i} z^{1-b} e^{\xi z} \oint_{C'} \frac{dt\, e^{-zt}}{(t-\xi)^{b-\alpha} (t-\xi+1)^{\alpha}} $$