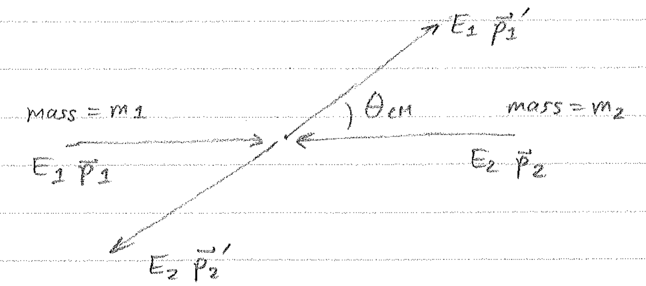
Center of Mass frame kinematics

$2 \to 2$ Elastic scattering



$\vec{p}_1 = (0, 0, 1)\,|\vec{p}|$ $\qquad \vec{p}_1{}' = (\sin\theta_{CM}, \ 0, \ \cos\theta_{CM})\,|\vec{p}|$

$\vec{p}_2 = (0, 0, -1)\,|\vec{p}|$ $\qquad \vec{p}_2{}' = (-\sin\theta_{CM}, \ 0, \ -\cos\theta_{CM})\,|\vec{p}|$

COM energy:

$E_{CM} = E_1 + E_2$

$= \dfrac{\vec{p}^2}{2\mu}$ ← reduced mass $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

COM frame defined by $\vec{p}_1 = -\vec{p}_2 \Rightarrow |\vec{p}_1| = |\vec{p}_2|$

Conservation of momentum:

$$\underbrace{\vec{p}_1 + \vec{p}_2}_{0} = \vec{p}_1{}' + \vec{p}_2{}'$$

$$0 = \vec{p}_1{}' + \vec{p}_2{}' \Rightarrow \vec{p}_1{}' = -\vec{p}_2{}' \Rightarrow |\vec{p}_1{}'| = |\vec{p}_2{}'|$$

Conservation of energy:

$$E_1 + E_2 = E_1{}' + E_2{}'$$

$$\frac{|\vec{p}|^2}{2m_1} + \frac{|\vec{p}|^2}{2m_2} = \frac{|\vec{p}\,'|^2}{2m_1} + \frac{|\vec{p}\,'|^2}{2m_2} \Rightarrow \boxed{|\vec{p}_1| = |\vec{p}_2| = |\vec{p}_1{}'| = |\vec{p}_2{}'|}$$

Magnitudes of all four momenta the same $\Rightarrow E_1 = E_1{}' \quad E_2 = E_2{}'$.

Response variable θ_{CM} is no longer tied to $E_1{}'$ or $E_2{}'$.

Momentum transfer: $\vec{q} = \vec{p}_1{}' - \vec{p}_1 = -(\vec{p}_2{}' - \vec{p}_2)$

$$\begin{aligned}\vec{q}^{\,2} &= \vec{p}_1^{\,2} + \vec{p}_1{}'^{\,2} - 2\vec{p}_1 \cdot \vec{p}_1{}' \\ &= 2|\vec{p}|^2 - 2|\vec{p}|^2 \cos\theta \\ &= 2\vec{p}^{\,2}(1 - \cos\theta) \\ &= 4\vec{p}^{\,2}\sin^2(\theta/2) \quad \text{or} \quad = 8\mu(E_1 + E_2)\sin^2\left(\tfrac{\theta}{2}\right)\end{aligned}$$