Normalization / completeness conventions (NR. QM)

	Basis	Normalization	Completeness
Free plane waves	position 3-vector:	$\langle \vec{x} \| \vec{x}' \rangle = \delta^{(3)}(\vec{x}-\vec{x}')$	$\int d^3x \, \|\vec{x}\rangle\langle\vec{x}\| = 1$
	wave-number 3-vector:	$\langle \vec{k} \| \vec{k}' \rangle = \delta^{(3)}(\vec{k}-\vec{k}')$	$\int d^3k \, \|\vec{k}\rangle\langle\vec{k}\| = 1$
free spherical waves	(radial) wave-number + angular mom:	$\langle k,l,m \| k',l',m' \rangle = \delta(k-k')\,\delta_{ll'}\,\delta_{mm'}$	$\int_0^\infty dk \sum_{l,m} \|k,l,m\rangle\langle k,l,m\| = 1$
	energy + angular mom:	$\langle E,l,m \| E',l',m' \rangle = \delta(E-E')\,\delta_{ll'}\,\delta_{mm'}$	$\int_0^\infty dE \sum_{l,m} \|E,l,m\rangle\langle E,l,m\| = 1$

Connecting formulae — in position/momentum/energy rep.

$$\langle \vec{x} | \vec{k} \rangle = \psi_{\vec{k}}(\vec{x}) = \frac{1}{\sqrt{(2\pi)^3}} e^{i\vec{k}\cdot\vec{x}}$$

Rayleigh expansion

$$= \underbrace{\frac{1}{\sqrt{(2\pi)^3}} 4\pi}_{\sqrt{2/\pi}} \sum_{l=0}^{\infty} i^l j_l(kr) \sum_{m=-l}^{l} Y_l^m(\Omega_{\vec{x}})\, Y_l^{*m}(\Omega_{\vec{k}})$$

derived from norm. of sph. Bessel function

$$\langle \vec{x} | k,l,m \rangle = \psi_{k,l,m}(\vec{x}) = i^l \sqrt{\frac{2}{\pi}}\, k\, j_l(kr)\, Y_l^m(\Omega_{\vec{x}}) = i^l \sqrt{\frac{2}{\pi}} \frac{1}{r} \hat{j}_l(kr)\, Y_l^m(\Omega_{\vec{x}})$$

$$= R^{free}_{k,l}(r)\, Y_l^m(\Omega_{\vec{x}}) \qquad = i^l \sqrt{\frac{2}{\pi}} \frac{1}{r} u_l^{free}(k,r)\, Y_l^m(\Omega_{\vec{x}})$$

definition of free radial functions

$$\boxed{\begin{aligned} u_l^{free}(k,r) &= \hat{j}_l(kr) \\ R^{free}_{k,l}(r) &= i^l \sqrt{\frac{2}{\pi}} \frac{1}{r} \hat{j}_l(kr) \end{aligned}}$$

$$\langle \vec{k}' | k,l,m \rangle = \frac{1}{k} \delta(k-k')\, Y_l^m(\Omega_{\vec{k}}) \quad ✓$$

Free spherical & plane waves

Replacing $k \to E$: $\quad |k,l,m\rangle = \sqrt{\frac{\hbar^2 k}{M}}\,|E,l,m\rangle \qquad E = \frac{\hbar^2 k^2}{2M}$

deduced by changing variables in completeness relation.

$$\langle \vec{x}|E,l,m\rangle = \psi_{E,l,m}(\vec{x}) = i^l \sqrt{\frac{2\hbar^2 k M}{\pi}}\, j_l(kr)\, Y_l^m(\Omega_{\vec{x}}) = i^l \sqrt{\frac{2\hbar^2 M}{\pi k}}\, \frac{1}{r}\, \hat{j}_l(kr)\, Y_l^m(\Omega_{\vec{x}})$$

$$\langle \vec{k}|E,l,m\rangle = \frac{\hbar}{\sqrt{Mk}}\, \delta(E_{\vec{k}} - E)\, Y_l^m(\Omega_{\vec{k}})$$

Note:

$$\delta^{(3)}(\vec{k}'-\vec{k}) = \frac{\hbar^2}{km}\,\delta(E'-E)\,\delta^{(2)}(\Omega_{k'}-\Omega_k)$$

$$\delta(E'-E) = \frac{m}{\hbar^2 k}\,\delta(k'-k)$$

$$dk = \frac{1}{\hbar}\sqrt{\frac{M}{2E}}\,dE$$

or

$$dE = \frac{\hbar^2 k}{M}\,dk$$

Orthonormality relations in position space

$$\int d^3x\; \psi^*_{\vec{k}}(\vec{x})\, \psi_{\vec{k}'}(\vec{x}) = \delta^{(3)}(\vec{k}-\vec{k}')$$

$$\int d^3x\; \psi^*_{k,l,m}(\vec{x})\, \psi_{k',l',m'}(\vec{x}) = \delta(k-k')\,\delta_{ll'}\,\delta_{mm'}$$

$$\int d^3x = \int dr\, r^2 \int d\Omega_{\vec{x}}$$

$$\left.\begin{aligned} &\int_0^\infty dr\; r^2 R^*_{k,l}(r)\, R_{k',l}(r) = \delta(k-k') \\ &\int_0^\infty dr\; u_l^*(k,r)\, u_l(k',r) = \frac{\pi}{2}\,\delta(k-k') \end{aligned}\right\} \text{same } l. \;\; !!$$

Completeness relations in (linear & angular) momentum space

$$\int d^3k\; \psi^*_{\vec{k}}(\vec{x})\, \psi_{\vec{k}}(\vec{x}') = \delta^{(3)}(\vec{x}-\vec{x}')$$

$$\int_0^\infty dk \sum_{l,m} \psi^*_{k,l,m}(\vec{x})\, \psi_{k,l,m}(\vec{x}') = \delta^{(3)}(\vec{x}-\vec{x}')$$

Stationary (spherical wave) scattering states:

- defined by analogy

$$\langle k, l, m | k', l', m' \rangle = \delta(k-k')\,\delta_{ll'}\,\delta_{mm'}$$

$$\langle E, l, m | E', l', m' \rangle = \delta(E-E')\,\delta_{ll'}\,\delta_{mm'}$$

<u>but</u>

$$\int_0^\infty dk \sum_{l,m} |k, l, m\rangle\langle k, l, m| = 1 - \begin{pmatrix}\text{bound}\\ \text{states}\end{pmatrix}$$

$$\int_0^\infty dE \sum_{l,m} |E, l, m\rangle\langle E, l, m| = 1 - \begin{pmatrix}\text{bound}\\ \text{states}\end{pmatrix}$$

<u>Position space wavefunctions</u>

$$\langle \vec{x} | \vec{k} \rangle = \frac{1}{(2\pi)^{3/2}} \frac{1}{kr} \sum_{l=0}^{\infty} (2l+1)\, i^l\, u_l(k,r)\, P_l(\cos\theta)$$

$$\langle \vec{x} | k, l, m \rangle = \psi_{k,l,m}(\vec{x}) = R_{k,l}(r)\, Y_l^m(\Omega_{\vec{x}}) = i^l \sqrt{\frac{2}{\pi}}\, \frac{1}{r}\, u_l(k,r)\, Y_l^m(\Omega_{\vec{x}})$$

$$\langle \vec{x} | E, l, m \rangle = \psi_{E,l,m}(\vec{x}) = R_{E,l}(r)\, Y_l^m(\Omega_{\vec{x}}) = i^l \sqrt{\frac{2\hbar M}{\pi k}}\, \frac{1}{r}\, u_l(k,r)\, Y_l^m(\Omega_{\vec{x}})$$

Orthornormality:

$$\int d^3x\, \psi^*_{k,l,m}(\vec{x})\, \psi_{k',l',m'}(\vec{x}) = \delta(k-k')\,\delta_{ll'}\,\delta_{mm'}$$

$$\int_0^\infty dr\, r^2\, R^*_{k,l}(r)\, R_{k',l}(r) = \delta(k-k')$$

$$\int_0^\infty dr\, u_l^*(k,r)\, u_l(k',r) = \frac{\pi}{2}\,\delta(k-k')$$

completeness:

$$\int_0^\infty dk \sum_{l,m} \psi^*_{k,l,m}(\vec{x})\, \psi_{k,l,m}(\vec{x}') = \delta^{(3)}(\vec{x}-\vec{x}')$$