Linear Algebra review

Definition: Unitary operator $\hat{U}$

A linear operator that maps the whole of $\mathcal{H}$ onto the whole of $\mathcal{H}$, and preserves the norm:

$$\|\hat{U}\varphi\| = \|\varphi\| \quad \text{for all } \varphi \in \mathcal{H}$$

Definition: Isometric operator $\hat{\Omega}$

A linear operator which is defined on the whole of $\mathcal{H}$ and preserves the norm:

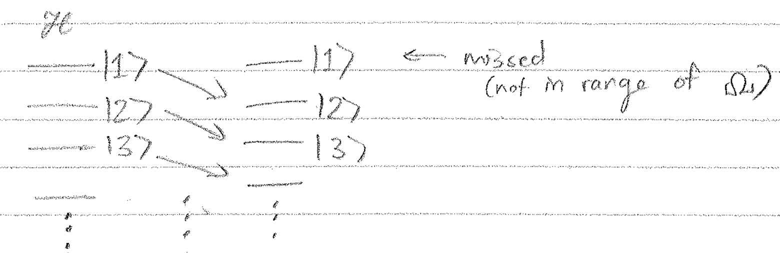
MAY NOT MAP ONTO WHOLE OF $\mathcal{H}$.

$$\|\Omega\varphi\| = \|\varphi\| \quad \text{for all } \varphi \in \mathcal{H}$$

On a finite-dimensional space, they are the same: Isometric operator $\Longleftrightarrow$ Unitary operator.

On an infinite-dimensional space, operators may be Isometric but not Unitary.

eg:



If Ω isometric, $\Omega^\dagger\Omega = 1$ (but $\Omega\Omega^\dagger \neq 1$)

Convergence of Vectors:

$|\psi_t\rangle$ converges to $|\psi\rangle$ if:

$$\|\psi_t - \psi\| \longrightarrow 0 \quad \text{as } t\to\infty \quad \text{(definition)}$$

[this formal def^{n} practically useless]

Useful to note: If $|\psi_t\rangle$ converges to $|\psi\rangle$, then its expansion coefficients converges to same (non-zero) numbers as that of $|\psi\rangle$.

$$|\psi_t\rangle = \sum_n c_n(t)|n\rangle \xrightarrow{t\to\infty} \sum_n c_n|n\rangle = |\psi\rangle$$

(then these two states become physically indistinguishable)

Notion of state indistinguishability

$|\psi\rangle$ is completely identified if we require the numbers $|\langle\phi|\psi\rangle|$ are known for all normalizable $|\phi\rangle$.

Now consider $|\langle\phi|\psi_t\rangle - \langle\phi|\psi\rangle| = |\langle\phi|(|\psi_t\rangle - |\psi\rangle)|$

schwarz inequality

$$\leq \underbrace{\|\phi\|}_{1} \, \|\psi_t - \psi\|$$

$$\Rightarrow |\langle\phi|\psi_t\rangle - \langle\phi|\psi\rangle| \leq \underbrace{\|\psi_t - \psi\|}$$

this number tends to zero as $t \to \infty$.

Therefore, by making t large enough, the difference between $\langle\phi|\psi_t\rangle$ and $\langle\phi|\psi\rangle$ for all normalized $|\phi\rangle$ becomes smaller that any prescribed ϵ.

say, detector resolution.

$\Rightarrow$ $|\psi_t\rangle$ & $|\psi\rangle$ become more & more indistinguishable as $t \to \infty$.

Cauchy test for convergence

The vector $|\psi_t\rangle$ whose convergence we wish to test will appear as an integral:

$$|\psi_t\rangle = \int_0^t d\tau \, |\phi_\tau\rangle$$

converges if $\quad \left\| \int_t^{t'} d\tau \, |\phi_\tau\rangle \right\| \xrightarrow{t,t'\to\infty} 0$

This is satisfied if

$$\int_t^{t'} d\tau \, \|\phi_\tau\| \xrightarrow{t,t'\to\infty} 0$$

And this is true if and only if

$$\int_0^{\infty} d\tau \, \|\phi_\tau\| = \text{finite number.}$$

Operator limits:

An operator A_t has a limit A if for every $|\psi\rangle$, the vector $A_t|\psi\rangle$ has a limit:

$$A_t|\psi\rangle \xrightarrow[t\to\infty]{} |\phi\rangle = A|\psi\rangle \qquad \text{for all } |\psi\rangle \in \mathcal{H}$$

[Notice: Defined in terms of vector convergence]

Careful: If $A_t \to A$, this does not mean $A_t^\dagger \to A^\dagger$
and if also $B_t \to B$, this does not mean $A_t B_t \to AB$

[In fact, the limit of a unitary operator is not necessarily unitary.
It is guaranteed only to be isometric]