Formal properties of Møller and Scattering operators

Intertwining Relation for Møller operators

$$\hat{H}\hat{\Omega}_{\pm} = \hat{\Omega}_{\pm}\hat{H}_0$$

Proof: Consider $e^{i\hat{H}\tau}\hat{\Omega}_{\pm}$

$$e^{i\hat{H}\tau}\Omega_{\pm} = e^{i\hat{H}\tau}\left[\lim_{t\to\mp\infty} e^{i\hat{H}t}e^{-i\hat{H}_0 t}\right]$$

$$= \lim_{t\to\mp\infty}\left[e^{i\hat{H}(t+\tau)}e^{-i\hat{H}_0 t}\right]$$

$$= \underbrace{\lim_{t\to\mp\infty}\left[e^{i\hat{H}(t+\tau)}e^{-i\hat{H}_0(t+\tau)}\right]}_{\Omega_{\pm}} e^{i\hat{H}_0\tau}$$

shift symm. is CRUCIAL!
NEED $t\to\infty$.

$$= \Omega_{\pm}\, e^{i\hat{H}_0\tau}$$

Now differentiate with respect to τ, and put $\tau=0$:

$$\left(i\hat{H}\, e^{i\hat{H}\tau}\right)\Omega_{\pm} = \Omega_{\pm}\left(i\hat{H}_0\, e^{i\hat{H}_0\tau}\right)$$

$$\hat{H}\,\Omega_{\pm} = \Omega_{\pm}\hat{H}_0 \quad ✓$$

Important consequence

If $|\psi_{\substack{in\\out}}\rangle \in \mathcal{S}$ is an e-state of $\hat{H}_0$ with energy E,

then $|\psi\rangle = \Omega_{\pm}|\psi_{\substack{in\\out}}\rangle \in \mathcal{R}$ is an e-state of $\hat{H}$ with same energy E:

proof

$$\hat{H}|\psi\rangle = \hat{H}\Omega_{\pm}|\psi_{\substack{in\\out}}\rangle = \Omega_{\pm}\hat{H}_0|\psi_{\substack{in\\out}}\rangle = E\left(\Omega_{\pm}|\psi_{\substack{in\\out}}\rangle\right) = E|\psi\rangle$$

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Therefore, the in/out state of energy E evolves to an actual orbit scattering state with the same energy.

Orthogonality theorem

Bound states are orthogonal to scattering states $\mathcal{B} \perp \mathcal{R}$

Assumptions:

I $V(r) = O\left(\frac{1}{r^{3+\epsilon}}\right)$ as $r \to \infty$

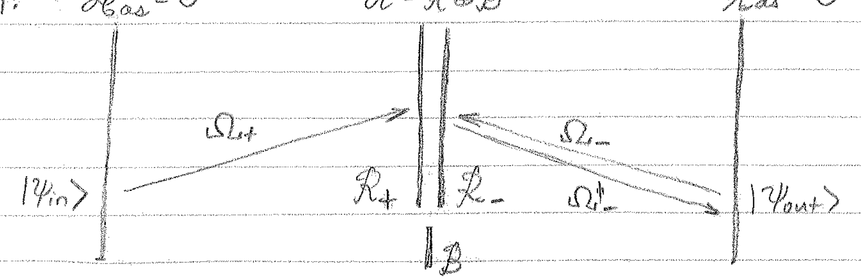
II $V(r) = O\left(\frac{1}{r^{3/2-\epsilon}}\right)$ as $r \to 0$

III $V(r) =$ continuous, except for finite number of discontinuities.

Asymptotic completeness

$$\underbrace{\left\{\begin{array}{l}\text{all states}\\ \text{with in-}\\ \text{asymptote}\end{array}\right\}}_{\mathcal{R}_+} = \underbrace{\left\{\begin{array}{l}\text{all states}\\ \text{with out-}\\ \text{asymptote}\end{array}\right\}}_{\mathcal{R}_-} = \left\{\begin{array}{l}\text{all states orthogonal}\\ \text{to the bound states}\end{array}\right\}$$

	Space of asymp. IN states	Space of actual orbits.	Space of asymp. out states.
Spanned by e-states of:	$\hat{H}_0$	$\hat{H}$	$\hat{H}_0$
Symbol:	$\mathcal{H}_{as} = \mathcal{S}$	$\mathcal{H} = \mathcal{R} \oplus \mathcal{B}$	$\mathcal{H}_{as} = \mathcal{S}$



NOTE!! Space of in-states and out-states are the __same__ space!

Scattering operator

$S = \Omega_-^\dagger \Omega_+$ $\qquad |\psi_{out}\rangle = \hat{S}\,|\psi_{in}\rangle$

with $|\psi_{out}\rangle, |\psi_{in}\rangle \in \mathcal{S}$