Interaction picture unitary time evolution operator:

$$\hat{U}_I = \hat{U}_0^\dagger(t_f, t_i)\ \hat{U}(t_f, t_i)$$

$$\hat{U}_I^\dagger = \hat{U}^\dagger(t_f, t_i)\ \hat{U}_0(t_f, t_i)$$

so $\Omega_- \coloneqq \lim_{t_f \to \infty} U^\dagger(t_f, 0)\, U_0(t_f, 0) = \lim_{t \to \infty} U_I^\dagger(t, 0)$

and $\Omega_+ \coloneqq \lim_{t_f \to -\infty} U_I^\dagger(t_f, 0) = \lim_{t_f \to +\infty} U_I^\dagger(-t_f, 0) = \lim_{t \to +\infty} U_I(0, -t)$

Then

$$\hat{\Omega}_-^\dagger \Omega_+ = \left[\lim_{t\to\infty} \hat{U}_I^\dagger(t,0)\right]^\dagger \lim_{t\to\infty} \hat{U}_I(0,-t)$$

incorrect set of manipulations $\downarrow$ just heuristics (~~final answer~~ is correct)

$$\stackrel{!}{\neq} \lim_{t\to\infty} U_I(t,0) \lim_{t\to\infty} \hat{U}_I(0,-t)$$

$$\stackrel{!}{\neq} \lim_{t\to\infty} \underbrace{U(t,0)\, U_I(0,-t)}_{U_I(t,-t)} = \lim_{t\to\infty} U_I(t,-t) \equiv \hat{S}$$

$\therefore$ Any method of calculating U_I should provide an expression for $\hat{S}$ in $t \to \infty$ limit.

e.g. time-dependent perturbation theory, applied at large times yields the Born series for $\hat{S}$.

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