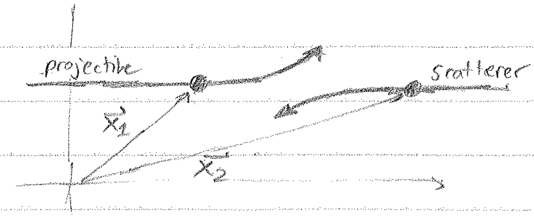
Reduction of two-body scattering to one-body potential scattering

- Let 1 and 2 be two distinguishable particles



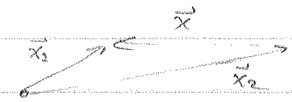
$$\hat{H}_{sys} = \frac{\hat{\vec{p}}_1^{\,2}}{2m_1} + \frac{\hat{\vec{p}}_2^{\,2}}{2m_2} + \hat{V}(x_1 - x_2)$$

Reduction from two-body problem to one-body problem:

- perform a canonical transformation on coordinates $(\vec{x}_1, \vec{p}_1)$ & $(\vec{x}_2 \vec{p}_2)$.

center-of-mass coordinates and relative coordinates.

$$\vec{X}_{com} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2} \qquad \vec{x} = \vec{x}_1 - \vec{x}_2$$



$$\vec{P}_{com} = \vec{p}_1 + \vec{p}_2 \qquad \vec{p} = \frac{m_2 \vec{p}_1 - m_1 \vec{p}_2}{m_1 + m_2} = \hbar \vec{k}$$

Abbreviate $M = m_1 + m_2$ = total mass of system

$m = \dfrac{m_1 m_2}{m_1 + m_2}$ = reduced mass of system.

So that

$$\hat{H}_{sys} = \underbrace{\frac{\vec{P}_{com}^{\,2}}{2M}}_{\text{com motion of system}} + \underbrace{\frac{\vec{p}^{\,2}}{2m} + V(\vec{x})}_{\text{relative motion of two particles.}} = \hat{H}_{com} + \hat{H}$$

Since $[\vec{P}_{com}, \vec{p}] = 0$, state of system specified by $|\vec{K}_{com}\rangle \otimes |\vec{k}\rangle$

So that
$$\langle \vec{k}_1' \vec{k}_2' | \hat{S}_{sys} | \vec{k}_1 \vec{k}_2 \rangle = \langle \vec{K}_{com}', \vec{k}' | \mathbb{1}_{com} \otimes \hat{S} | \vec{K}_{com}, \vec{k} \rangle$$
$$= \delta^{(3)}(\vec{K}_{com}' - \vec{K}) \underbrace{\langle \vec{k}' | \hat{S} | \vec{k} \rangle}_{\text{one-particle } \hat{S}\text{-matrix.}}$$

Substitute cluster later decomposition for $\langle \vec{k}' | \hat{S} | \vec{k} \rangle$ to obtain:

$$\boxed{\begin{aligned} \langle \vec{k}_1' \vec{k}_2' | \hat{S}_{sys} | \vec{k}_1 \vec{k}_2 \rangle = {} & \delta^{(3)}(\vec{K}_{com}' - \vec{K}_{com})\, \delta^{(3)}(\vec{k}' - \vec{k}) \\ & + \delta^{(3)}(\vec{K}_{com}' - \vec{K}_{com}) \frac{i\hbar^2}{2\pi m} \delta(E_{\vec{k}'} - E_{\vec{k}})\, f(\vec{k}' \leftarrow \vec{k}). \end{aligned}}$$

Note that mass m appearing in non-relativistic potential scattering is the <u>reduced mass</u>: $m = \dfrac{m_1 m_2}{m_1 + m_2}$.