Operator formulation of non-relativistic scattering theory

S-matrix element:

Transition probability of asymptotic in-state to asymptotic out-state:

$$w(\text{out} \leftarrow \text{in}) = \langle \text{out} | \hat{S} | \text{in} \rangle$$

Properties of problem can give us form for $\hat{S}$:

- conservation of energy: $[\hat{H}_0, \hat{S}] = 0$ (all that is needed is commutation with $\hat{H}_0$)

$$\Rightarrow \langle \vec{k}' | \hat{S} | \vec{k} \rangle = (2\pi)\,\delta(E_{\vec{k}'} - E_{\vec{k}}) \left(\text{stuff}\right)$$

plane waves: $e^{i\vec{k}\cdot\vec{x}}$; (2π): convention

The T-matrix element:

This property of $\hat{S}$ follows from the more general (cluster decomposition theorem.) → For short range potentials, we can expect: $\hat{S} = \hat{1} + i\hat{T}$ (i: convenience.)

$$\langle \vec{k}' | \hat{S} | \vec{k} \rangle = \langle \vec{k}' | \hat{1} + i\hat{T} | \vec{k} \rangle$$

$$= \langle \vec{k}' | \vec{k} \rangle + i \langle \vec{k}' | \hat{T} | \vec{k} \rangle$$

$$= \delta^{(3)}(\vec{k}' - \vec{k}) - 2\pi i\, \delta(E_{\vec{k}'} - E_{\vec{k}})\, t(\vec{k}' \leftarrow \vec{k})$$

t: This defines only elements of $\hat{T}$ for which $E_{\vec{k}'} = E_{\vec{k}}$ (hence, this is the on-shell $\hat{T}$-matrix)

Define scattering amplitude by:

$$f(\vec{k}' \leftarrow \vec{k}) = \underbrace{-(2\pi)^2 m}_{\text{convention}}\, t(\vec{k}' \leftarrow \vec{k})$$

so that

$$\boxed{\begin{aligned} \langle \vec{k}' | \hat{S} | \vec{k} \rangle &= \delta^{(3)}(\vec{k}' - \vec{k}) - 2\pi i\, \delta(E_{\vec{k}'} - E_{\vec{k}}) \frac{-\hbar^2}{(2\pi)^2 m} f(\vec{k}' \leftarrow \vec{k}) \\ &= \delta^{(3)}(\vec{k}' - \vec{k}) + \frac{i\hbar^2}{2\pi m}\, \delta(E_{\vec{k}'} - E_{\vec{k}})\, f(\vec{k}' \leftarrow \vec{k}) \end{aligned}}$$

Reason for making all these definitions:

Elements of $\hat{S}$ and $\hat{T}$ are highly singular functions of $\vec{k}$ & $\vec{k}'$ due to $\delta(E_{\vec{k}} - E_{\vec{k}'})$.

However, $f(\vec{k}' \leftarrow \vec{k})$ is well-behaved, and is mathematically more manageable.