Optical Theorem

- Direct consequence of unitarity:

$$\hat{S}^\dagger \hat{S} = \hat{\mathbb{1}}$$

cluster decomp: $$(\hat{\mathbb{1}} - i\hat{T}^\dagger)(\mathbb{1} + i\hat{T}) = \hat{\mathbb{1}}$$

$$\mathbb{1} + i\hat{T} - i\hat{T} + \hat{T}^\dagger \hat{T} = \mathbb{1}$$

$$-i(\hat{T} - \hat{T}^\dagger) = \hat{T}^\dagger \hat{T}$$

Take matrix elements between plane waves $\langle \vec{k}'|$ and $|\vec{k}\rangle$:

$$-i\left(\langle \vec{k}'|\hat{T}|\vec{k}\rangle - \langle \vec{k}'|\hat{T}^\dagger|\vec{k}\rangle\right) = \langle \vec{k}'|\hat{T}^\dagger \hat{T}|\vec{k}\rangle$$

insert resolution of identity: $\int d^3\vec{k}'' \, |\vec{k}''\rangle\langle \vec{k}''|$

$$-i\left(\langle \vec{k}'|\hat{T}|\vec{k}\rangle - \langle \vec{k}'|\hat{T}^\dagger|\vec{k}\rangle\right) = \int d^3\vec{k}'' \, \langle \vec{k}'|\hat{T}^\dagger|\vec{k}''\rangle\langle \vec{k}''|\hat{T}|\vec{k}\rangle$$

Put in expression for scattering amplitude: $\langle \vec{k}'|\hat{T}|\vec{k}\rangle = \frac{\hbar^2}{2\pi m}\delta(E_{\vec{k}'} - E_{\vec{k}}) f(\vec{k}' \leftarrow \vec{k})$

$\langle \vec{k}'|\hat{T}^\dagger|\vec{k}\rangle = \langle \vec{k}|\hat{T}|\vec{k}'\rangle^*$

$$-i\frac{\hbar^2}{2\pi m}\delta(E_{k'} - E_k)\left(f(\vec{k}' \leftarrow \vec{k}) - f^*(\vec{k} \leftarrow \vec{k}')\right)$$

$$= \left(\frac{\hbar^2}{2\pi m}\right)^2 \int d^3\vec{k}'' \, \delta(E_{\vec{k}'} - E_{\vec{k}''})\delta(E_{\vec{k}''} - E_{\vec{k}}) f^*(\vec{k}'' \leftarrow \vec{k}') f(\vec{k}'' \leftarrow \vec{k})$$

RHS: Write $\delta(E_{\vec{k}'} - E_{\vec{k}''})\delta(E_{\vec{k}''} - E_{\vec{k}}) = \delta(E_{\vec{k}'} - E_{\vec{k}''})\,\delta\left(\frac{\hbar^2 \vec{k}''^2}{2m} - \frac{\hbar^2 \vec{k}^2}{2m}\right)$

$$= \delta(E_{\vec{k}'} - E_{\vec{k}''}) \times \frac{m}{\hbar^2|\vec{k}''|}\underbrace{\delta(|\vec{k}''| - |\vec{k}|)}$$

and $\int d^3\vec{k}'' = \int d\Omega_{\vec{k}''} \, d|\vec{k}''| \, |\vec{k}''|^2$, and integrate over $d|\vec{k}''|$,

fixing magnitude: $|\vec{k}''| \to |\vec{k}|$.

but not vector $\vec{k}'' \not\to \vec{k}$

$$-\frac{i\hbar^2}{2\pi m}\,\delta(E_{\vec k'}-E_{\vec k})\left(f(\vec k'\leftarrow\vec k)-f^*(\vec k\leftarrow\vec k')\right)$$

$$=\left(\frac{\hbar^2}{2\pi m}\right)^2\int d\Omega_{\vec k''}\,\frac{|\vec k|^2 m}{\hbar^2|\vec k|}\,\delta(E_{\vec k'}-E_{\vec k})\,f^*(\vec k''\leftarrow\vec k')f(\vec k''\leftarrow\vec k)$$

$$=\frac{\hbar^2|\vec k|}{4\pi^2 m}\,\delta(E_{\vec k'}-E_{\vec k})\int d\Omega_{\vec k''}\,f^*(\vec k''\leftarrow\vec k')\,f(\vec k''\leftarrow\vec k)$$

cancel $\delta(E_{k'}-E_k)$ and $\frac{\hbar^2}{2\pi m}$

$$\frac{-i}{2}\left(f(\vec k'\leftarrow\vec k)-f^*(\vec k\leftarrow\vec k')\right)=\frac{|\vec k|}{4\pi}\int d\Omega_{\vec k''}\,f^*(\vec k''\leftarrow\vec k')\,f(\vec k''\leftarrow\vec k) \qquad (*)$$

If potential is spherically symmetric, (implying P & T invariance), then in LHS:

$$f^*(\vec k\leftarrow\vec k')\underset{[\text{P invariance}]}{=}f^*(-\vec k\leftarrow-\vec k')\underset{[\text{T invariance}]}{=}f^*(\vec k'\leftarrow\vec k)$$

so that we can write LHS as $\frac{-i}{2}\left(f(\vec k'\leftarrow\vec k)-f(\vec k''\leftarrow\vec k)\right)=\operatorname{Im} f(\vec k'\leftarrow\vec k)$

$$\underline{\underline{\operatorname{Im} f(\vec k'\leftarrow\vec k)=\frac{|\vec k|}{4\pi}\int d\Omega_{\vec k''}\,f^*(\vec k''\leftarrow\vec k)\,f(\vec k''\leftarrow\vec k)}}$$

(generalized optical theorem for central potentials [P & T invariant]

Specialize to case of forward scattering: $\vec k'=\vec k$. Then (*) becomes:

$$\operatorname{Im} f(\vec k\leftarrow\vec k)=\frac{|\vec k|}{4\pi}\underbrace{\int d\Omega_{\vec k''}\left|f(\vec k''\leftarrow\vec k)\right|^2}_{\sigma^{\text{elastic}}_{\text{total}}}$$

$$\boxed{\operatorname{Im} f(\vec k\leftarrow\vec k)=\frac{|\vec k|}{4\pi}\,\sigma_{tot}}$$

No P, T invariance required.