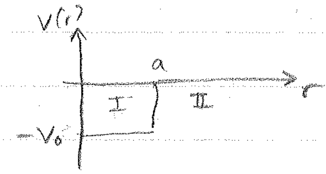
Continuum Solutions for the potential Well



Look for solutions with $E > 0$.

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} + \frac{2m}{\hbar^2}\left(E - V(r)\right)\right] R(r) = 0.$$

Define (wavenumbers) $\frac{2m}{\hbar^2}\left(E - V(r)\right) = \begin{cases} \frac{2m}{\hbar^2}(E + V_0) = q^2 \equiv k^2 + \xi^2/a^2 & \text{inside} \\ \frac{2m}{\hbar^2} E = k^2 & \text{outside} \end{cases}$

Inside well, want solution to be regular at origin, and outside well, can be a superposition of j_ℓ and n_ℓ ← STATES NOT NORMALIZABLE

$$R_I(r) = A\, j_\ell(qr) \qquad r < a$$

$$R_{II}(r) = B\, j_\ell(kr) + C\, n_\ell(kr) \qquad r > a$$

Matching conditions:

As before, we have derivatives of logs!

$$\frac{\partial}{\partial r} \ln R_I(a) = \frac{\partial}{\partial r} \ln R_{II}(a)$$

$$q\,\frac{\cancel{A}\, j_\ell'(qa)}{\cancel{A}\, j_\ell(qa)} = k\,\frac{B\, j_\ell'(ka) + C\, n_\ell'(ka)}{B\, j_\ell(ka) + C\, n_\ell(ka)} = k\,\frac{j_\ell'(ka) + \frac{C}{B} n_\ell'(ka)}{j_\ell(ka) + \frac{C}{B} n_\ell(ka)}$$

solve for C/B:

$$\frac{C}{B} = \frac{-q\, j_\ell'(qa)\, j_\ell(ka) + k\, j_\ell(qa)\, j_\ell'(ka)}{q\, j_\ell'(qa)\, n_\ell(ka) - k\, j_\ell(qa)\, n_\ell'(ka)}$$

In the region far away $r \gg a$, asymptotically (use $R_{II}(r)$)

$$R(r) \simeq \frac{B}{kr}\sin\left(kr - \frac{\ell\pi}{2}\right) - \frac{C}{kr}\cos\left(kr - \frac{\ell\pi}{2}\right)$$

$$\simeq \frac{B}{kr}\left[\sin\left(kr - \frac{\ell\pi}{2}\right) \underbrace{- \frac{C}{B}}_{} \cos\left(kr - \frac{\ell\pi}{2}\right)\right]$$

Define $\tan\delta_\ell(k) \equiv -\frac{C}{B}$

Then,

$$R(r) \simeq \frac{B}{\cos \delta_\ell(k)} \frac{1}{kr} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell(k)\right)$$

recall: $k = \sqrt{2mE}/\hbar$

For s-wave $(\ell = 0)$ it is possible to obtain a simple expression for $\delta_{\ell=0}(k)$

$$q \frac{j_\ell'(qa)}{j_\ell(qa)} = k \frac{B\, j_\ell'(ka) + C n_\ell'(ka)}{B\, j_\ell(ka) + C n_\ell(ka)}$$

$$j_0'(x) = \frac{1}{x}\left(\cos x - \frac{1}{x}\sin x\right)$$

$$n_0'(x) = \frac{1}{x}\left(\sin x + \frac{1}{x}\cos x\right)$$

Smoothness condition

continuity condition

group

$$q \frac{\cancel{\frac{1}{qa}}\left(\cos qa - \frac{1}{qa}\sin qa\right)}{\cancel{\frac{1}{qa}}\sin qa} = k \frac{\frac{B}{\cancel{ka}}\left(\cos ka - \frac{1}{ka}\sin ka\right) + \frac{C}{\cancel{ka}}\left(\sin ka + \frac{1}{ka}\cos ka\right)}{\frac{B}{\cancel{ka}}\sin ka - \frac{C}{\cancel{ka}}\cos ka}$$

$$q\left(\cot qa - \frac{1}{qa}\right) = k\left(\frac{B\cos ka + C\sin ka}{B\sin ka - C\cos ka} + \frac{1}{ka}\underbrace{\frac{-B\sin ka + C\cos ka}{B\sin ka - C\cos ka}}_{=-1}\right)$$

$$q\cot qa - \cancel{\frac{1}{a}} = k\frac{B\cos ka + C\sin ka}{B\sin ka - C\cos ka} - \cancel{\frac{1}{a}}$$

$$= k\frac{\cos ka + \frac{C}{B}\sin ka}{\sin ka - \frac{C}{B}\cos ka} = k\frac{\cos ka - \tan\delta_0(k)\sin ka}{\sin ka + \tan\delta_0(k)\cos ka}$$

$\cos a \cos b - \sin a \sin b = \cos(a+b)$
$\cos a \sin b + \sin a \cos b = \sin(a+b)$

$$= k\frac{\cancel{\frac{1}{\cos\delta_0}}\cos(ka + \delta_0)}{\cancel{\frac{1}{\cos\delta_0}}\sin(ka+\delta_0)} = k\cot(ka + \delta_0)$$

$$\frac{1}{q}\tan qa = \frac{1}{k}\tan(ka + \delta_0)$$

$$\boxed{\delta_0(k) = \tan^{-1}\left(\frac{k}{q}\tan qa\right) - ka}$$

$$q = +\sqrt{\frac{2m}{\hbar^2}(E + V_0)} = \sqrt{k^2 + \xi^2/a^2}$$

$$k = +\sqrt{\frac{2m}{\hbar^2}E}$$