Symmetries + Conservation laws

Translation invariance:

Translation operator: $\hat{D}(\vec{\alpha}) = e^{-i\vec{\alpha}\cdot\hat{\vec{P}}}$

$\langle 2|\,\hat{\mathbb{S}}\,|1\rangle = \langle 2|\,\hat{D}^{-1}\,\hat{\mathbb{S}}\,\hat{D}\,|1\rangle$ [translation inv.]

$= \langle 2|\,e^{i\vec{\alpha}\cdot\vec{P}}\,\hat{\mathbb{S}}\,e^{-i\vec{\alpha}\cdot\vec{P}}\,|1\rangle$

$= e^{i\vec{\alpha}\cdot(\vec{K}'_{tot}-\vec{K}_{tot})}\langle 2|\,\hat{\mathbb{S}}\,|1\rangle$

set LHS = RHS. since $\vec{K}'_{tot} = \vec{K}_{tot} \Rightarrow \hat{\mathbb{S}}$ prop to $\delta(\vec{K}'_{tot} - \vec{K}_{tot})$

$\Rightarrow$ Seperate out COM motion: $\hat{\mathbb{S}} = \mathbb{1}_{CM} \otimes \hat{S}$

leads to $\delta(\vec{K}'_{tot} - \vec{K}_{tot})$ — conventional $\hat{S}$ matrix.

acts on $\mathcal{H}_{CM}$ — acts on $\mathcal{H}_{rel}$

Rotational invariance:

Rotation operator: $\hat{R}(\vec{\alpha}) = e^{-i\vec{\alpha}\cdot\vec{J}_{tot}}$

$$\vec{J}_{tot} = \vec{J}_1 + \vec{J}_2$$
$$= \vec{x}_1 \times \vec{P}_1 + S_1 + \vec{x}_2 \times \vec{P}_2 + S_2$$
$$= \vec{X}_{CM} \times \vec{P}_{CM} + \underbrace{\vec{x} \times \vec{P}_{rel} + \vec{S}_1 + \vec{S}_2}$$
$$= J_{CM} + \vec{J}$$

overall angular momentum — internal angular momentum.

acts on $\mathcal{H}_{CM}$ — acts on $\mathcal{H}_{rel}$

Rotation operator relevant for $\mathcal{H}_{rel}$:

$$\hat{R}(\vec{\alpha}) = e^{-i\alpha(\vec{L} + \underbrace{\vec{S}_1 + \vec{S}_2})}$$
$$= \hat{R}_L(\vec{\alpha})\,\hat{R}_S(\vec{\alpha})$$

If $\hat{V}(\vec{x})$ is spherically symmetric,

$\hat{H}$ is invariant under rotations $\Rightarrow$ $\hat{S}$ is invariant.

$$\Rightarrow \langle \vec{k}' | \hat{S} | \vec{k} \rangle = \langle \vec{k}' | e^{i\alpha \vec{J}} \hat{S} e^{-i\alpha \vec{J}} | \vec{k} \rangle$$

$$= \langle \vec{k}'_{Rot} | \hat{S} | \vec{k}_{Rot} \rangle$$

Substitute S-matrix element in terms of scattering amplitude in LHS and RHS

$$\cancel{\delta^{(3)}(\vec{k}' - \vec{k})} + \frac{i\hbar^2}{2\pi m} \cancel{\delta(E_{\vec{k}'} - E_{\vec{k}})} f(\vec{k}' \leftarrow \vec{k})$$

$$= \cancel{\delta^{(3)}(\vec{k}'_{Rot} - \vec{k}_{Rot})} + \frac{i\hbar^2}{2\pi m} \cancel{\delta(E_{\vec{k}'} - E_{\vec{k}})} f(\vec{k}'_{Rot} \leftarrow \vec{k}_{Rot})$$

$$\boxed{f(\vec{k}' \leftarrow \vec{k}) = f(\vec{k}'_{Rot} \leftarrow \vec{k}_{Rot})}$$

Count DOF:

2 particles $\equiv$ 6 components $(\vec{k}'_1, \vec{k}'_2)$

$\Downarrow$ seperate COM motion

$\equiv$ 5 components

$\Downarrow$ rotational invariance

$\equiv$ 2 components $(E_{\vec{k}}, \theta)$

$$\boxed{f(\vec{k}' \leftarrow \vec{k}) = f(E_{\vec{k}}, \theta)}$$