Partial-wave Series

- Very convenient way to incorporate S-matrix unitarity.

$\hat{S}$ commutes with $\hat{L}$ and $\hat{L}_z$ $\Rightarrow$ basis $\{|E,l,m\rangle\}$ diagonalizes $\hat{S}$:

eigenvalue.

$$\langle E',l',m'|\hat{S}|E,l,m\rangle = \langle E',l',m|E,l,m\rangle \, S_l(E)$$

$$= \delta(E'-E)\,\delta_{ll'}\,\delta_{mm'}\, e^{2i\delta_l(E)}$$

$\hat{S}^\dagger = \hat{S}^{-1}$ (unitary)

Note: $\delta_l(E) \equiv$ "phase shift"

Connecting relationship: $\hat{S} = 1 + i\hat{T}$

$$\langle \vec{k}'|\hat{S}|\vec{k}\rangle = \langle \vec{k}'|\vec{k}\rangle + \frac{i\hbar^2}{2\pi m}\delta(E'-E)\,f(E,\theta)$$

$$\langle \vec{k}'|(\hat{S}-1)|\vec{k}\rangle = \frac{i\hbar^2}{2\pi m}\delta(E'-E)\,f(E,\theta)$$

insert $\int dE \sum_{l\,m} |E,l,m\rangle\langle E,l,m|$

$$\int dE \sum_{l,m} \langle \vec{k}'|(\hat{S}-1)|E,l,m\rangle\langle E,l,m|\vec{k}\rangle = \frac{i\hbar^2}{2\pi m}\delta(E'-E)\,f(E,\theta)$$

$$\int dE \sum_{l,m} \langle \vec{k}'|E,l,m\rangle\langle E,l,m|\vec{k}\rangle\,(S_l(E)-1) =$$

$$\int dE \sum_{l,m} \frac{\hbar}{\sqrt{mk}}\frac{\hbar}{\sqrt{mk'}}\,\delta(E_{\vec{k}'}-E)\,\delta(E-E_{\vec{k}})\,Y_l^m(\Omega_{\vec{k}'})\,Y_l^{*m}(\Omega_{\vec{k}})\,(S_l(E)-1) =$$

integrate over E

$$\frac{\hbar^2}{mk}\delta(E_{\vec{k}'}-E)\sum_l \underbrace{\sum_m Y_l^m(\Omega_{\vec{k}'})\,Y_l^{*m}(\Omega_{\vec{k}})}_{\frac{2l+1}{4\pi}P_l(\cos\Theta_{\vec{k}-\vec{k}'})}\,(S_l(E)-1) = \frac{i\hbar^2}{2\pi m}\delta(E'-E)\,f(E,\theta)$$

$\equiv$ scattering angle θ

$$\frac{\hbar^2}{mk}\cancel{\delta(E_{\vec{k}'}-E)}\sum_l \frac{2l+1}{4\pi}P_l(\cos\theta)\,(S_l(E)-1) = \frac{i\hbar^2}{2\pi m}\cancel{\delta(E'-E)}\,f(E,\theta)$$

$$\boxed{\sum_{l=0}^{\infty}(2l+1)\,\frac{1}{2ik}\,(S_l(E)-1)\,P_l(\cos\theta) = f(E,\theta)}$$

Match against partial wave decomposition:

$$f(E,\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, f_\ell(E)\, P_\ell(\cos\theta)$$

partial wave ampl.

inverse relation:

$$f_\ell(E) = \frac{1}{2}\int_{-1}^{1} d(\cos\theta)\, f(E,\theta)\, P_\ell(\cos\theta)$$

$$\Rightarrow \quad f_\ell(E) = \frac{1}{2ik}\left(S_\ell(E) - 1\right) = \frac{1}{2ik}\left(e^{2i\delta_\ell(E)} - 1\right) = \frac{1}{k} e^{i\delta_\ell(E)} \sin\delta_\ell(E)$$