Unitarity circle and Argand diagram

Recall unitarity of S-matrix: $\hat{S}^\dagger \hat{S} = \hat{\mathbb{1}}$.

If $\hat{S}$ were diagonal,
(true in angular momentum basis for spherical potential)

each eigenvalue s_ℓ is just a complex number
(forget for the moment that they are functions of $E = \frac{\hbar^2 k^2}{2m}$)

With the cluster decomposition, $S_\ell = 1 + i f_\ell$,
What does it imply for f_ℓ given $|S_\ell|^2 = 1$?

$$|S_\ell|^2 = (1 + i f_\ell)(1 - i f_\ell^*)$$
$$= \cancel{1} + \underbrace{i(f_\ell - f_\ell^*)}_{2i\,\mathrm{Im} f_\ell} + |f_\ell|^2 = \cancel{1}$$

NOTE! Depending on how f_ℓ is defined, there may be additional factors of 8π or even kinematic factors.

For example:
N.R. scattering, $S_\ell = 1 + 2ik f_\ell$
relativistic, $S_\ell = 1 + \frac{8\pi i}{v} A_\ell$

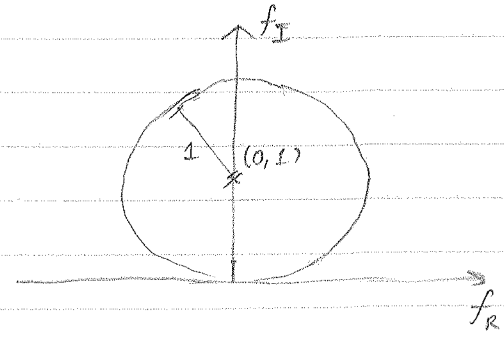
$$-2\,\mathrm{Im} f_\ell + |f_\ell|^2 = 0$$

Drop ℓ label, and write $\mathrm{Re} f = f_R$ & $\mathrm{Im} f = f_I$.

$$-2 f_I + f_R^2 + f_I^2 = 0$$
complete the square

$$f_R^2 + (f_I - 1)^2 - 1 = 0$$
or $$\boxed{f_R^2 + (f_I - 1)^2 = 1}$$

On the complex plane, this is a circle of radius $= 1$
and centered at $(0, 1)$.



Thus, f_ℓ must lie somewhere on this circle — and since it is a function of $E = \frac{\hbar^2 k^2}{2m}$, it must lie on this circle for all E.