Quantum Scattering cross section

Stationary Wave function:
$$\psi(x) = \frac{1}{\sqrt{\text{Vol}}}\Big(\underbrace{e^{i\vec{k}\cdot\vec{x}}}_{\text{Incoming}} + \underbrace{f(\theta,\phi)\frac{e^{ikr}}{r}}_{\text{outgoing}}\Big)$$

$\text{vol.} \equiv (2\pi)^3$

$$\text{Flux} = \frac{\#\text{ of particles}}{(\text{unit area})(\text{unit time})} \equiv \text{current density}$$
[FOR ONE PARTICLE]

Heuristic only! Need time-dep. scattering theory for this.

Definition of quantum cross section:

$$d\sigma = \frac{1}{\left(\frac{\#\text{ of inc. particles}}{(\text{unit area})(\text{unit time})}\right)}\left(\frac{\#\text{ of particles scattered}}{(\text{unit time})}\right) \text{ into } \left(\begin{matrix}\text{detector}\\ \text{element}\end{matrix}\ d\Omega\right)$$

Multiply/divide by unit area.

$$= \frac{1}{\text{inc. Flux}}(\text{scattered flux})\ dA$$

Recall, $\vec{J} = \frac{\hbar}{2im}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)$

Flux for incoming wave: $\psi_{inc} = \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{\text{Vol.}}}$

$$\vec{J}_{inc} = \frac{\hbar}{2im}\frac{1}{\text{Vol.}}\left(e^{-i\vec{k}\cdot\vec{x}}(i\vec{k})e^{i\vec{k}\cdot\vec{x}} - \text{c.c.}\right)$$

$$= \frac{\hbar}{2im}\frac{1}{\text{Vol.}}2i\vec{k}\ \cancel{e^{-i\vec{k}\cdot x}}\ \cancel{e^{+i\vec{k}\cdot x}}$$

$$= \frac{\hbar\vec{k}}{m}\frac{1}{\text{Vol.}} = \frac{\vec{p}}{m}\frac{1}{\text{Vol.}} = \frac{\vec{v}}{\text{Vol.}}$$

$\vec{k}$ is typically oriented along z-direction, so $\vec{J}_{inc}$ is only along z-direction.

Flux for outgoing (scattered) wave: $\psi_{out} = \frac{1}{\sqrt{Vol.}} f(\theta,\phi) \frac{e^{ikr}}{r}$

- interested in flux at location of detector $r = R \gg \frac{1}{k}$ $\quad$ R = location of detector.

Need: $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\hat{\phi}\frac{\partial}{\partial\phi}$

$$\vec{J}_{out} = \frac{\hbar}{2im}\frac{1}{Vol.}\left[f^*(\theta,\phi)\frac{e^{-ikr}}{r}\left(\hat{r}\frac{\partial}{\partial r} + O\left(\tfrac{1}{r}\right)\right) f(\theta,\phi)\frac{e^{ikr}}{r} - c.c.\right]$$

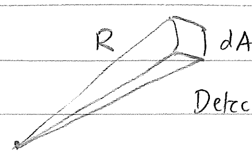
$$= \frac{\hbar}{2im}\frac{1}{Vol.}\left[f^*(\theta,\phi)\frac{e^{-ikr}}{r}\hat{r} f(\theta,\phi)\left(\frac{ik}{r} - \frac{1}{r^2}\right)e^{ikr} + O\left(\tfrac{1}{r^2}\right) - c.c.\right]$$

add.

$$= \frac{\hbar}{2im}\frac{1}{Vol.}\left[|f(\theta,\phi)|^2 \frac{1}{r^2}\hat{r}\; 2ik\right]$$

$$= \frac{\hbar k}{m}\frac{1}{Vol.}\frac{|f(\theta,\phi)|^2}{r^2}\hat{r}$$

Finally, $\quad dA = R^2 d\Omega$



So that:

$$d\sigma = \frac{1}{\cancel{\left(\frac{\hbar k}{m}\frac{1}{Vol.}\right)}}\left(\cancel{\frac{\hbar k}{m}\frac{1}{Vol.}}\frac{|f(\theta,\phi)|^2}{\cancel{R^2}}\right)\cancel{R^2} d\Omega$$

$$d\sigma = |f(\theta,\phi)|^2 d\Omega$$

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2}$$