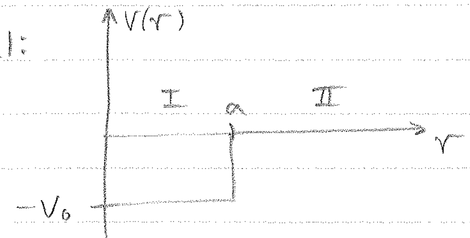
Phase Shifts for Scattering from a Spherical Potential Well

Potential:



$$V(r) = \begin{cases} -V_0, & r < a \\ 0, & r > a \end{cases}$$

Recall, inside the well (Region I: $r<a$), the radial wavefunction has the form:

$$R_I(r) = A\, j_\ell(qr) \qquad q = \frac{2m}{\hbar^2}(E+V_0), \quad E>0$$

(must use scattering states only)

For phase shifts, need $\gamma_\ell = \frac{\partial}{\partial r} \ln R_I(r) \Big|_{r=a}$ ← Logarithmic derivative of the interior solution at $r=a$.

$$= q\,\frac{j_\ell'(qa)}{j_\ell(qa)}$$

Then:
$$\cot\delta_\ell = \frac{q\, n_\ell'(qa) - k\,\frac{j_\ell'(ka)}{j_\ell(ka)}\, n_\ell(ka)}{q\, j_\ell'(qa) - k\,\frac{j_\ell'(ka)}{j_\ell(ka)}\, j_\ell(ka)} \qquad k^2 = \frac{2m}{\hbar^2}E$$

$$= \frac{k\, j_\ell(qa)\, n_\ell'(ka) - q\, j_\ell'(qa)\, n_\ell(ka)}{k\, j_\ell(qa)\, j_\ell'(ka) - q\, j_\ell'(qa)\, j_\ell(ka)}$$

(Precisely what we got when calculating $\frac{-C}{B}$ when matching boundary conditions)

Phase shifts are written in terms of external wavenumber k
Eliminate q in favor of k.

$$q = k^2 + \frac{2mV_0}{\hbar^2} = k^2 + \left(\frac{\zeta}{a}\right)^2, \quad \text{where } \zeta^2 = \frac{2ma^2V_0}{\hbar^2}$$

characterizes strength of potential.

So, multiply num. and denom. by a, and combine: $qa = \sqrt{(ka)^2+\zeta^2}$

$$\boxed{\cot\delta_\ell(k) = \frac{ka\, j_\ell\left(\sqrt{(ka)^2+\zeta^2}\right) n_\ell'(ka) - \sqrt{(ka)^2+\zeta^2}\; j_\ell'\left(\sqrt{(ka)^2+\zeta^2}\right) n_\ell(ka)}{ka\, j_\ell\left(\sqrt{(ka)^2+\zeta^2}\right) j_\ell'(ka) - \sqrt{(ka)^2+\zeta^2}\; j_\ell'\left(\sqrt{(ka)^2+\zeta^2}\right) j_\ell(ka)}}$$

Consider $l=0$:

$$\begin{cases} j_0(x) = \dfrac{\sin x}{x} & j_0'(x) = \dfrac{\cos x}{x} - \dfrac{\sin x}{x^2} \\ n_0(x) = \dfrac{-\cos x}{x} & n_0'(x) = \dfrac{\sin x}{x} + \dfrac{\cos x}{x^2} \end{cases}$$

$$\tan\delta_0(k) = \frac{k\, j_0(qa)\, j_l'(ka) - q\, j_0'(qa)\, j_0(ka)}{k\, j_0(qa)\, n_0'(ka) - q\, j_0'(qa)\, n_0(ka)}$$

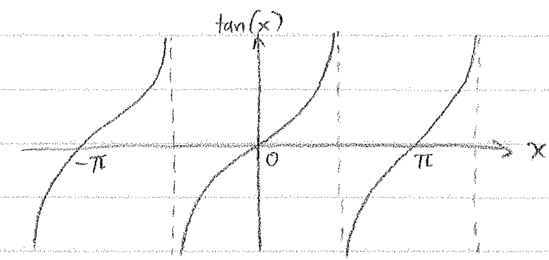
$$\delta_0(k) = \underbrace{-ka}_{\text{hard sphere scattering contribution}} + \underbrace{\tan^{-1}\left(\frac{k}{q}\tan qa\right)}_{\text{part unique to the potential.}}$$

[see "Continuum Solutions for Potential Well" notes]

— Such decomposition can be made, in general.
— later.

Consider behavior of δ_0 at low energies $ka \ll 1$ $\left(q = \sqrt{k^2 + \frac{2m}{\hbar^2}V_0}\right)$.
If the potential is weak, $\frac{2m}{\hbar^2}V_0 a^2 \ll 1$, then we also have $qa \ll 1$.

Expand around $k \approx 0$.



$$\tan^{-1}(\# k) = \underset{\substack{\uparrow \\ \text{Labels roots}}}{n\pi} + \# k + O(k^2)$$

So, $$\delta_0(k) = -ka + n\pi + \frac{\tan qa}{q}k + O(k^2)$$

↑ Ambiguity: Phase shifts mod π are equal

But, recall:

Absolute def$^{\underline{n}}$ of phase shifts: $\delta_0 \to 0$ when $\frac{2m}{\hbar^2}V \to 0$.

$\Rightarrow$ fixes n: $q \to k$ as $\frac{2m}{\hbar^2}V \to 0$

$$\begin{aligned} \delta_0(k) &= -ka + n\pi + \frac{\tan(ka)}{\cancel{k}}\cancel{k} + O(k^2) \\ &= \cancel{-ka} + n\pi + \cancel{ka} + O(k^2) \\ &= n\pi \quad \Rightarrow \quad n\pi = 0 \end{aligned}$$

Hence, absolute phase shift is $$\boxed{\delta_0(k) = ka\left(\frac{\tan qa}{qa} - 1\right) + O(k^2)}$$

valid for small V_0 and k

Appendix –

S-matrix eigenvalues and partial amplitudes for spherical well potential.

$$S_\ell(k) = \frac{-2k\, j_\ell'(ka) + 2\gamma_\ell\, j_\ell(ka)}{k\, h_\ell^{(1)\prime}(ka) - \gamma_\ell\, j_\ell(ka)}$$

$q = \sqrt{k^2 + \zeta^2 a^2}$

and $\zeta = \sqrt{2ma^2 V_0/\hbar}$

where $\gamma_\ell = q \dfrac{j_\ell'(qa)}{j_\ell(qa)}$ for spherical well potential.

so

$$S_\ell(k) - 1 = \frac{-2k\, j_\ell'(ka) + 2q \dfrac{j_\ell'(qa)}{j_\ell(qa)} j_\ell(ka)}{k\, h_\ell^{(1)\prime}(ka) - q \dfrac{j_\ell'(qa)}{j_\ell(qa)} h_\ell^{(1)}(qa)}$$

$$= \frac{2\left[-k\, j_\ell'(ka) j_\ell(qa) + q\, j_\ell'(qa)\, j_\ell(ka)\right]}{k\, h_\ell^{(1)\prime}(ka)\, j_\ell(qa) - q\, j_\ell'(qa)\, h_\ell^{(1)}(ka)}$$

Then partial-wave amplitude is:

$$f_\ell(k) = \frac{1}{2ik}\left(S_\ell(k) - 1\right)$$

so

$$|f_\ell(k)|^2 = \frac{1}{4k^2}\left|S_\ell(k) - 1\right|^2 = \frac{1}{k^2}\left|\frac{-k\, j_\ell'(ka) j_\ell(qa) + q\, j_\ell'(qa)\, j_\ell(ka)}{k\, h_\ell^{(1)\prime}(ka)\, j_\ell(qa) - q\, j_\ell'(qa)\, h_\ell^{(1)}(ka)}\right|^2$$

and partial wave cross sections are:

$$\sigma_\ell(k) = 4\pi\,(2\ell+1)\,|f_\ell(k)|^2$$

$$= \frac{4\pi\,(2\ell+1)}{4k^2}\left|S_\ell(k) - 1\right|^2$$

and total cross section is:

$$\sigma(k) = \sum_{\ell=0}^{\infty} \sigma_\ell(k)$$