Unitarity: Each partial wave satisfies its own unitarity relation.

$\hat{S}^\dagger \hat{S} = \mathbb{1}$

$\rightarrow$ angular momentum basis ($\hat{S} \equiv$ diagonal $= e^{2i\delta_\ell(k)}$)

Then using $S_\ell = 1 + 2ikf_\ell$,

$$S_\ell^* S_\ell = \cancel{1} = (1 - 2ikf_\ell^*)(1 + 2ikf_\ell)$$

$$= \cancel{1} + 2ik\underbrace{(f_\ell - f_\ell^*)}_{2i\,\mathrm{Im}\, f_\ell} + 4k^2 f_\ell^* f_\ell$$

$$0 = -4k\,\mathrm{Im}\, f_\ell + 4k^2 f_\ell^* f_\ell$$

$$\boxed{\mathrm{Im}\, f_\ell = k f_\ell^* f_\ell \equiv k|f_\ell(k)|^2}$$

ASIDE: Unitarity relation for reduced partial wave (for N/D method):

$$B_\ell(k) \equiv \frac{1}{k^{2\ell}} f_\ell(k)$$

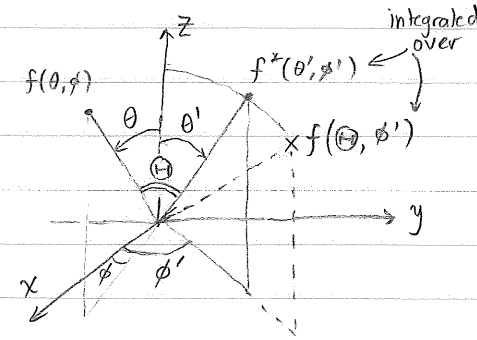
$$\Rightarrow k^{2\ell}\, \mathrm{Im}\, B_\ell(k) = k\, k^{4\ell} |B_\ell(k)|^2$$

$$\mathrm{Im}\, B_\ell(k) = k^{2\ell+1} |B_\ell(k)|^2$$

Unitarity Relation — general case of the optical theorem

function of θ'

$$\mathrm{Im}\, f_k(\theta,\phi) \equiv \mathrm{Im}\, f_k(\theta) = \frac{k}{4\pi}\int d\Omega'\, f_k^*(\theta',\phi')\, f_k(\Theta,\phi')$$



PROOF: In RHS, substitute

$$(*) \qquad f_k(\theta,\phi) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\, e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta)$$

So, RHS becomes:

$$\text{RHS} = \frac{k}{4\pi}\,\frac{1}{k^2}\sum_{l,l'}(2l+1)(2l'+1)\, e^{i(\delta_l-\delta_{l'})}\sin\delta_l\sin\delta_{l'}\int d\Omega'\, P_l(\cos\Theta)\, P_{l'}(\cos\theta')$$

Use: $$P_l(\cos\Theta) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$ ADDITION THEOREM FOR SPHERICAL HARMONICS

$$P_{l'}(\cos\theta') = \sqrt{\frac{4\pi}{2l'+1}}\; Y_{l',0}(\theta',\phi') \quad (\text{indep of } \phi')$$ DEFINITION OF SPH. HARM.

$$\text{RHS} = \frac{1}{\cancel{4\pi} k}\sum_{l,l'}(\cancel{2l+1})(2l'+1)\, e^{i(\delta_l-\delta_{l'})}\sin\delta_l\sin\delta_{l'}$$
$$\times\int d\Omega'\,\frac{\cancel{4\pi}}{\cancel{2l+1}}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)\sqrt{\frac{4\pi}{2l'+1}}\; Y_{l',0}(\theta',\phi')$$

Pull out of integral

$$= \frac{1}{k}\sum_{l,l'}(2l'+1)\, e^{i(\delta_l-\delta_{l'})}\sin\delta_l\sin\delta_{l'}\, Y_{lm}(\theta,\phi)$$
$$\times\sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l'+1}}\;\underbrace{\int d\Omega'\, Y^*_{lm}(\theta',\phi')\, Y_{l',0}(\theta',\phi')}_{=\delta_{ll'}\delta_{m0}}$$

ORTHONORMALITY RELATION

Sum over m, fixing $m\to 0$, & sum over l', fixing $l'\to l$.

$$= \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l\,\underbrace{Y_{l0}(\theta,\phi)}_{\sqrt{\frac{2l+1}{4\pi}}\,P_l(\cos\theta)}\sqrt{\frac{4\pi}{2l+1}}$$

$$= \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l\, P_l(\cos\theta) = \mathrm{Im}\, f_k(\theta,\phi) = \text{LHS} \quad ✓$$

Notice, in the case $\theta = 0$, we have $\Theta = \theta'$ under the integral:

$$\operatorname{Im} f_k(\theta = 0) = \frac{k}{4\pi} \int d\Omega' \, f_k^*(\theta', \phi') \, f_k(\theta', \phi')$$

$$= \frac{k}{4\pi} \int d\Omega' \, |f_k(\theta', \phi')|^2 \equiv \frac{k}{4\pi} \sigma_{Tot},$$

we recover the <u>Optical Theorem</u>, as a special case.