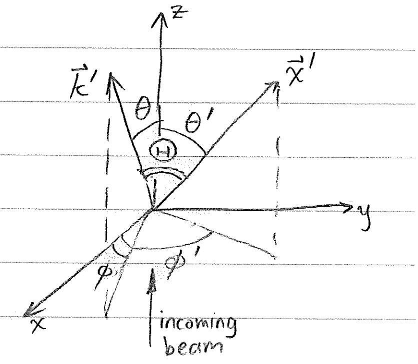
Partial Wave Scattering Amplitude, $f_\ell(k)$ in terms of $V(r)$

Defined by $\boxed{f_{\vec{k}}(\theta,\phi) = \sum_{\ell=0}^{\infty} (2\ell+1)\, f_\ell(k)\, P_\ell(\cos\theta)}$



Start from: $f_{\vec{k}}(\theta,\phi) = \dfrac{-m}{2\pi\hbar^2}\displaystyle\int d^3\vec{x}'\, \underbrace{e^{-i\vec{k}'\cdot\vec{x}'}}_{①}\, V(\vec{x}')\, \underbrace{\psi_{\vec{k}}(\vec{x}')}_{②}$

Write $e^{-i\vec{k}'\cdot\vec{x}'}$ & $\psi_{\vec{k}}(\vec{x}')$ as series in partial waves:

① $$e^{i\vec{k}'\cdot\vec{x}'} = \sum_{\ell=0}^{\infty} i^\ell\, \cancel{(2\ell+1)}\, j_\ell(\overset{|\vec{k}|=|\vec{k}'|}{k'x'})\, \frac{4\pi}{\cancel{2\ell+1}} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi')$$

$$= 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^\ell j_\ell(kx')\, Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi')$$

use parity $\vec{x}' \to -\vec{x}'$
$Y_{\ell m}(\theta,\phi) \to (-1)^\ell Y_{\ell m}(\theta,\phi)$

$$e^{-i\vec{k}'\cdot\vec{x}'} = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (-i)^\ell j_\ell(kx')\, Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi')$$

AND

② $$\psi_{\vec{k}}(\vec{x}') = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\, R_\ell(kx')\, P_\ell(\cos\theta')$$

$R_\ell \equiv$ solution to Radial Schrödinger equation

So, the scattering amplitude becomes:

$$f_{\vec{k}}(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\int d^3x' \left[4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (-i)^\ell j_\ell(kx')\, Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi')\right] V(\vec{x}')$$
$$\times \left[\sum_{\ell'=0}^{\infty} i^{\ell'} (2\ell'+1)\, R_{\ell'}(kx')\, P_{\ell'}(\cos\theta)\right]$$

to spherical polars $\vec{x}' \to (r', \theta', \phi')$

$$= \frac{-m}{2\pi\hbar^2}\int_0^{2\pi} d\phi' \int_{-1}^{1} d\cos\theta' \int dr' (r')^2 \left[4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (-i)^\ell j_\ell(kr')\, Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi')\right]$$
$$\times V(r') \left[\sum_{\ell'=0}^{\infty} i^{\ell'} (2\ell'+1)\, R_{\ell'}(kr')\, P_{\ell'}(\cos\theta')\right]$$

Integrate over θ', ϕ':

$$\int_0^{2\pi} d\phi' \int_{-1}^{1} d(\cos\theta')\, Y^*_{\ell m}(\cos\theta')\, P_{\ell'}(\cos\theta')$$

$Y_{\ell, m=0}$ indep of ϕ; can put anything in here: choose ϕ'.

$$= \int_0^{2\pi} d\phi' \int_{-1}^{1} d(\cos\theta)\, Y^*_{\ell m}(\cos\theta') \sqrt{\frac{4\pi}{2\ell'+1}}\, Y_{\ell' 0}(\theta', \phi')$$

$$= \sqrt{\frac{4\pi}{2\ell+1}}\, \delta_{\ell\ell'}\delta_{m0}$$

After summing over ℓ' & m,

$$f_k(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\int dr'(r')^2\left[4\pi\sum_{\ell=0}^{\infty}\cancel{\sqrt{\frac{4\pi}{2\ell+1}}}\,\cancel{(-i)^\ell}\, j_\ell(kr')\,\overbrace{Y_{\ell m=0}(\theta,\phi)}^{\sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta)}\right]V(r')$$

$$\times\, \cancel{i^\ell}\,(2\ell+1)\,R_\ell(kr')$$

$$= \sum_{\ell=0}^{\infty}(2\ell+1)\left[\frac{-2m}{\hbar^2}\int dr'(r')^2 j_\ell(kr')\,R_\ell(kr')\,V(r')\right]P_\ell(\cos\theta)$$

Compare with definition of $f_\ell(k)$, (dropping primes)

$$\boxed{f_\ell(k) = \frac{-2m}{\hbar^2}\int_0^\infty dr\, r^2\, j_\ell(kr)\,R_\ell(kr)\,V(r)}$$

$$\equiv -\int dr\, \frac{1}{k^2}\, kr\, j_\ell(kr)\;\, kr\, R_\ell(kr)\;\, \frac{2m}{\hbar^2}V(r)$$

$$= -\frac{1}{k^2}\int_0^\infty dr\;\, \hat{j}_\ell(kr)\;\, U(r)\;\, u_\ell(kr)$$

$$= \frac{-1}{k^2 f_\ell(k)}\int_0^\infty dr\;\, \hat{j}_\ell(kr)\;\, U(r)\;\, \phi_\ell(kr)$$

regular solution

Jost function

Riccati-Bessel function.