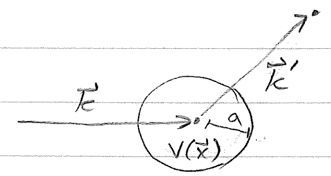
Scattering Phase Shifts

– Assume short range potential (vanishes for $|\vec{x}| > a$) $a \equiv$ characteristic range of potential

[results valid for weaker assumption $V(\vec{x}) < |\vec{x}|^{-2}$ for large r]

For $|\vec{x}| > a$ (where wavefunctions behave like free wavefunctions),



$|\vec{k}| = |\vec{k}'|$
(elastic scattering)

$$R_\ell^{\text{outside}}(kx) = N_\ell \Big(h_\ell^{(2)}(kx) + \underbrace{S_\ell(k)}_{\text{scattering amplitude}} \, h_\ell^{(1)}(kx) \Big)$$

Inside, $|\vec{x}| < a$, where the potential acts, the radial wavefunctions for each partial wave $R_\ell^{\text{inside}}(kx)$ must be solved for, either analytically or numerically. But once that is done, we may proceed to match wavefunctions and their derivatives at $x = a$ to get $S_\ell(k)$.

– convenient to differentiate logs:

Logarithmic derivative of the interior solution →

$$\underbrace{\frac{\partial}{\partial x} \ln R_\ell^{\text{inside}}(kx) \Big|_{x=a}}_{\text{denote} \equiv \gamma_\ell(k)} = \frac{\partial}{\partial x} \ln R_\ell^{\text{outside}}(kx) \Big|_{\ell = a}$$

$$= \frac{\frac{\partial}{\partial x}\left[h_\ell^{(2)}(kx) + S_\ell(k)\, h_\ell^{(1)}(kx) \right]}{h_\ell^{(2)}(kx) + S_\ell(k)\, h_\ell^{(1)}(kx)} \Bigg|_{x=a}$$

Solving for $S_\ell(k) - 1$ (lengthy algebra – use Mathematica)

$$S_\ell(k) - 1 = \frac{-\frac{\partial}{\partial x}\left(h_\ell^{(1)}(kx) + h_\ell^{(2)\prime}(kx) \right) + \gamma_\ell \left(h_\ell^{(1)}(kx) + h_\ell^{(2)}(kx) \right)}{\frac{\partial}{\partial x} h_\ell^{(1)}(kx) - \gamma_\ell \; h_\ell^{(1)}(kx)} \Bigg|_{x=a}$$

But, $h_\ell^{(1)}(kx) + h_\ell^{(2)}(kx) = 2 j_\ell(kx)$

$$S_\ell(k) \equiv e^{2 i \delta_\ell(k)} = \frac{2\left[-\frac{\partial}{\partial x} j_\ell(kx) + \gamma_\ell \, j_\ell(kx) \right]}{\frac{\partial}{\partial x} h_\ell^{(1)}(kx) - \gamma_\ell \, h_\ell^{(1)}(kx)} \Bigg|_{x=a} + 1$$

Exercise: Show that

$$\cot \delta_\ell(k) = \frac{\frac{\partial}{\partial x} n_\ell(kx) - \gamma_\ell \, n_\ell(kx)}{\frac{\partial}{\partial x} j_\ell(kx) - \gamma_\ell \, j_\ell(kx)} \Bigg|_{x=a}$$

Solution:

$$e^{2i\delta_\ell} = \frac{\text{num.}}{\text{den.}} + 1$$

$$2i\delta_\ell = \ln\left(\frac{\text{num}}{\text{den.}} + 1\right)$$

$$\delta_\ell = \frac{-i}{2}\ln\left(\frac{\text{num}}{\text{den.}} + 1\right)$$

$$\cot\delta_\ell = \cot\frac{-i}{2}\ln\left(\frac{\text{num}}{\text{den}} + 1\right)$$

$$= i\coth\left(\frac{1}{2}\ln\left(\frac{\text{num}}{\text{den}} + 1\right)\right)$$

use $\coth\ln(x^{1/2}) = \frac{1+x}{-1+x}$

$$= i\left[\frac{1+\frac{\text{num}}{\text{den}}+1}{\cancel{-1}+\frac{\text{num}}{\text{den}}\cancel{+1}}\right]$$

$$= i\left(1 + 2\frac{\text{den}}{\text{num}}\right)$$

$$= i\left(1 + \frac{\frac{\partial}{\partial x}h^{(1)}(kx) - \gamma_\ell h_\ell^{(1)}(kx)}{-\frac{\partial}{\partial x}j_\ell(kx) + \gamma_\ell j_\ell(kx)}\right)\Bigg|_{x=a}$$

$$= i\left(\frac{-\frac{\partial}{\partial x}j_\ell(kx) + \gamma_\ell(kx) + \frac{\partial}{\partial x}h^{(1)}(kx) - \gamma_\ell h^{(1)}(kx)}{-\frac{\partial}{\partial x}j_\ell(kx) + \gamma_\ell j_\ell(kx)}\right)\Bigg|_{x=a}$$

Simplify using $h_\ell = j_\ell + i n_\ell \Rightarrow h_\ell - j_\ell = i n_\ell$.

$$\boxed{\cot\delta_\ell(k) = \frac{\frac{\partial}{\partial x}n_\ell(kx) - \gamma_\ell n_\ell(kx)}{\frac{\partial}{\partial x}j_\ell(kx) - \gamma_\ell j_\ell(kx)}\Bigg|_{x=a}} = \frac{k n_\ell'(ka) - \gamma_\ell n_\ell(ka)}{k j_\ell'(ka) - \gamma_\ell j_\ell(ka)}$$

Example: Scattering off a solid sphere of radius a.

Inside, the wavefunctions vanish, so $\frac{\partial}{\partial x}\ln R_\ell^{\text{inside}}(kx) = \gamma_\ell \to \infty$.

$$\Rightarrow \cot\delta_\ell(k) = \frac{-\cancel{\gamma_\ell}n_\ell(kx)}{-\cancel{\gamma_\ell}j_\ell(kx)}\Bigg|_{x=a} = \frac{n_\ell(ka)}{j_\ell(ka)}$$

or $\boxed{\delta_\ell^{HS}(k) = \cot^{-1}\left(\frac{n_\ell(ka)}{j_\ell(ka)}\right)}$ for hard sphere.