Jost Function for Spherical Potential Well

$$f_\ell(k) = 1 + \frac{1}{k}\int_0^\infty dr\, \hat{h}^+_\ell(kr)\, U(r)\, \phi_\ell(k,r)$$

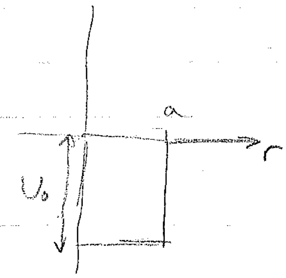
$[U] \equiv (\text{inverse length})^2$

$$U(r) = \begin{cases} -U_0, & r<a \\ 0, & r>a \end{cases}$$

$U_0 = \frac{2m}{\hbar^2} V_0$

$U_0 a^2 = \frac{2ma^2}{\hbar^2} V_0 \equiv \zeta$

Regular function for spherical potential well



To solve: $\left(\frac{\partial^2}{\partial r^2} - \frac{\ell(\ell+1)}{r^2} + k^2 - U(r)\right) \phi_\ell(r) = 0$

Solutions: $\overset{\text{outside}}{B\, \hat{j}_\ell(kr)}$ $\quad \overset{\text{inside}}{A(k)\hat{j}_\ell(\sqrt{k^2+U_0}\; r)}$

Only need solution inside — get $A(k)$ by demanding $A(k)\,\hat{j}_\ell(\sqrt{k^2+U_0}\, r) \xrightarrow{r\to 0} \hat{j}_\ell(kr)$

$$\phi_\ell(r\to 0) = A(k)\frac{2^\ell \ell!}{(2\ell+1)!}\left(\sqrt{k^2+U_0}\; r\right)^{\ell+1} = \frac{2^\ell \ell!}{(2\ell+1)!}(kr)^{\ell+1}$$

$$A(k)\left(\sqrt{k^2+U_0}\right)^{\ell+1} = k^{\ell+1}$$

$$A(k) = \left(\frac{k}{\sqrt{k^2+U_0}}\right)^{\ell+1} \Rightarrow \boxed{\phi_\ell(k) = \left(\frac{k}{\sqrt{k^2+U_0}}\right)^{\ell+1} \hat{j}_\ell(\sqrt{k^2+U_0}\; r)}$$

$$f_\ell(k) = 1 + \frac{1}{k}\int_0^a dr\, \hat{h}^+_\ell(kr)\,(-U_0)\left(\frac{k}{\sqrt{k^2+U_0}}\right)^{\ell+1} \hat{j}_\ell(\sqrt{k^2+U_0}\; r)$$

$$= 1 - U_0 \frac{k^\ell}{\sqrt{k^2+U_0}^{\,\ell+1}} \int_0^a dr\, \hat{h}^+_\ell(kr)\, \hat{j}_\ell(\sqrt{k^2+U_0}\; r)$$

Use Gradshteyn + Ryzhik ET II 367(26) (page 665) 7th Ed.

$$\hat{h}^+_\ell(kr) = i(kr)\, h^{(1)}_\ell(kr)$$
$$= i(kr)\left(j_\ell(kr) + i\, n_\ell(kr)\right)$$
$$= i(kr)\sqrt{\frac{\pi}{2kr}}\left(J_{\ell+1/2}(kr) + i\, Y_{\ell+1/2}(kr)\right)$$
$$= i\sqrt{\frac{\pi}{2}kr}\; H_{\ell+1/2}(kr) = i\sqrt{\frac{\pi}{2}kr}\;\frac{2}{\pi}\, i^{-\ell-3/2} K_{\ell+1/2}(-ikr)$$
$$= i^{-\ell-1/2}\sqrt{\frac{2}{\pi}kr}\; K_{\ell+1/2}(-ikr)$$

$$\hat{j}_\ell(k'r) = (k'r)\, j_\ell(k'r)$$
$$= (k'r)\sqrt{\frac{\pi}{2k'r}}\; J_{\ell+1/2}(k'r)$$
$$= \sqrt{\frac{\pi}{2}k'r}\; J_{\ell+1/2}(k'r)$$

$K_\alpha(-ix) = \frac{\pi}{2} i^{\alpha+1} H^{(1)}_\alpha(x)$

$H^{(1)}_\alpha(x) = \sqrt{\frac{2x}{\pi}}\, h^{(1)}_{\alpha-1/2}(x)$

$K_\alpha(-ix) = \sqrt{\frac{\pi}{2}x}\; i^{\alpha+1} h^{(1)}_{\alpha-1/2}(x)$

$$= 1 - U_0 \frac{k^\ell}{(k')^{\ell+1}}\, \underbrace{i^{-\ell-1/2}\sqrt{kk'}}\int_0^a dr\; r\, K_{\ell+1/2}(-ikr)\, J_{\ell+1/2}(k'r)$$

$$\int_0^a dr\, r\, J_{l+1/2}(k'r)\, K_{l+1/2}(-ikr)$$

$$= \Big(\underbrace{(-ik)^2}_{-k^2} + \underbrace{k'^2}_{k^2+U_0}\Big)^{-1}\left[\left(\frac{k'}{-ik}\right)^{l+1/2} + k'a\, J_{l+3/2}(k'a)\, K_{l+1/2}(-ika) - (-ika)\, J_{l+1/2}(k'a)\, K_{l+3/2}(-ika)\right]$$

$$= \frac{1}{U_0}\left[\left(\frac{\sqrt{k^2+U_0}}{-ik}\right)^{l+1/2} + k'a\sqrt{\frac{2k'a}{\pi}}\, j_{l+1}(k'a)\sqrt{\frac{\pi}{2}ka}\; i^{l+3/2}\, h_l^{(1)}(ka) + ika\sqrt{\frac{2}{\pi}k'a}\; j_l(k'a)\sqrt{\frac{\pi}{2}ka}\; i^{l+5/2}\, h_{l+1}^{(1)}(ka)\right]$$

$$= \frac{1}{U_0}\left[i(i)^{l+1/2}\left(\frac{\sqrt{k^2+U_0}}{k}\right)^{l+1/2} + i^{l+3/2}\sqrt{kk'}\, a\, k'a\, j_{l+1}(k'a)\, h_l^{(1)}(ka) + i^{l+7/2}\sqrt{kk'}\, a\, ka\, j_l(k'a)\, h_{l+1}^{(1)}(ka)\right]$$

So, multiplying by $i^{-l-1/2}\sqrt{kk'}$ to get $\int_0^a dr\, \hat{h}_l^+(kr)\, \hat{j}_l(\sqrt{k^2+U_0}\, r)$

$$= \frac{1}{U_0}\left[(i)^{l+1/2}\, i^{-l-1/2}\sqrt{kk'}\left(\frac{\sqrt{k^2+U_0}}{k}\right)^{l+1/2} + i^{l+3/2}\, i^{-l-1/2}(\sqrt{kk'})^2\, a k'a\, j_{l+1}(k'a)\, h_l^{(1)}(ka) + i^{l+7/2}\, i^{-l-1/2}(\sqrt{kk'})^2\, aka\, j_l(k'a)\, h_{l+1}^{(1)}(ka)\right]$$

$$= \frac{1}{U_0}\left[\frac{(\sqrt{k^2+U_0})^{l+1}}{k^l} + i\, kk'\, ak'a\, j_{l+1}(k'a)\, h_l^{(1)}(ka) + i^3\, kk'\, \underline{aka}\, j_l(k'a)\, h_{l+1}^{(1)}(ka)\right]$$

$$= \frac{1}{U_0}\left[\frac{\sqrt{k^2+U_0}^{\,l+1}}{k^l} + k'\,\hat{j}_{l+1}(k'a)\,\hat{h}_l^+(ka) - k\,\hat{j}_l(k'a)\,\hat{h}_{l+1}^+(ka)\right]$$

$\Rightarrow$

$$f_l(k) = 1 - U_0\frac{k^l}{\sqrt{k^2+U_0}^{\,l+1}}\frac{1}{U_0}\left[\frac{\sqrt{k^2+U_0}^{\,l+1}}{k^l} + k'\,\hat{j}_{l+1}(k'a)\,\hat{h}_l^+(ka) - k\,\hat{j}_l(k'a)\,\hat{h}_{l+1}^+(ka)\right]$$

$$= \frac{k^l}{\sqrt{k^2+U_0}^{\,l+1}}\left[k\,\hat{h}_{l+1}^+(ka)\,\hat{j}_l(\sqrt{k^2+U_0}\,a) - \sqrt{k^2+U_0}\;\hat{j}_{l+1}(\sqrt{k^2+U_0}\,a)\,\hat{h}_l^+(ka)\right]$$

$$= \frac{(ka)^l}{\sqrt{(ka)^2+\zeta^2}^{\,l+1}}\left[(ka)\,\hat{h}_{l+1}^+(ka)\,\hat{j}_l\big(\sqrt{(ka)^2+\zeta^2}\big) - \sqrt{(ka)^2+\cancel{U_0}\,\zeta^2}\;\hat{j}_{l+1}\big(\sqrt{(ka)^2+\zeta^2}\big)\,\hat{h}_l^+(ka)\right]$$