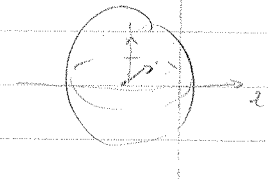
Total cross section in Eikonal approximation:

$$\frac{d\sigma}{d\Omega} = |f_k(\theta,\phi)|^2$$



$$\sigma = \int d\Omega \; |f_k(\theta,\phi)|^2$$

$$= \int d\Omega \; \frac{k^2}{4\pi^2} \left| \int d^2\vec{b} \; e^{-i\vec{q}\cdot\vec{b}} \; \Gamma(\vec{b}) \right|^2$$

Instead, obtain imaginary part of forward amplitude. $\vec{q} = 0$.

$$f_k(0,0) = \frac{ik}{2\pi} \int d^2\vec{b} \; \Gamma(\vec{b}) \qquad r = \sqrt{|\vec{b}|^2 + z^2}$$

$$= \frac{ik}{2\pi} \int d^2\vec{b} \left[1 - e^{\frac{1}{2ik}\int_{-\infty}^{\infty} dz \, U(\vec{b},z)} \right]$$

If $U(r) = \begin{cases} -\frac{2m}{\hbar^2} V_0 \, , & r < a \quad \text{or} \quad \sqrt{|\vec{b}|^2 + z^2} < a \\ 0 \, , & r > 0, \quad \sqrt{|\vec{b}|^2 + z^2} > a \end{cases}$ $\qquad z = \sqrt{a^2 - |\vec{b}|^2}$

Then $\displaystyle\int_{-\sqrt{a^2-b^2}}^{\sqrt{a^2-b^2}} dz \; -\frac{2m}{\hbar^2} V_0 = \begin{cases} -\frac{2m}{\hbar^2} V_0 \times 2\sqrt{a^2-b^2} & b < a \\ 0 & b > 0 \end{cases}$

$$f_k(0,0) = \frac{ik}{2\pi} \int d^2\vec{b} \left[1 - e^{\frac{1}{2ik} \frac{-4m}{\hbar^2} \sqrt{a^2-b^2} \begin{cases} V_0, & |\vec{b}| < a \\ 0 & |\vec{b}| > a \end{cases}} \right.$$

Each term is divergent.
But can split integration range.

put $d^2 b \to \int d\phi_b \, db \; b$

$$= \frac{ik}{2\pi} \int_0^{2\pi} d\phi \left[\int_0^a db \; b \left(1 - e^{\frac{2im}{\hbar^2 k}\sqrt{a^2-b^2}\, V_0} \right) + \underbrace{\int_a^{\infty} db \; b \left(1 - e^{\frac{1}{2ik} 0} \right)}_{=0} \right]$$

$$f_k(0) = \frac{ik}{2\pi}\underbrace{\int_0^{2\pi} d\phi}_{2\pi}\int_0^a db\, b\left[1-\exp\left(\frac{2imV_0}{\hbar^2 k}\sqrt{a^2-b^2}\right)\right]$$

$$= ik\left[\frac{a^2}{2} - \int_0^a db\, b\, \exp\left(\frac{2imV_0}{\hbar^2 k}\sqrt{a^2-b^2}\right)\right]$$

$$= ik\left[\frac{a^2}{2} + \frac{\hbar^4 k^2}{4m^2V_0^2}\left(1+\left(-1+\frac{2imV_0 a}{\hbar^2 k}\right)e^{\frac{2imV_0 a}{\hbar^2 k}}\right)\right]$$

recall:

$$\frac{2ma^2V_0}{\hbar^2} = \zeta^2 \qquad V_0 = \frac{\hbar^2\zeta^2}{2a^2 m}$$

$$= ik\left[\frac{a^2}{2} + \frac{\cancel{\hbar^4}k^2}{\cancel{4m^2}}\,\frac{\cancel{4}a^4\cancel{m^2}}{\cancel{\hbar^4}\zeta^4}\left(1+\left(-1+\frac{i\zeta^2}{ak}\right)e^{i\zeta^2/ak}\right)\right]$$

$$= ik\left[\frac{a^2}{2} + \frac{k^2a^4}{\zeta^4}\left(1+\left(-1+\frac{i\zeta^2}{ak}\right)e^{i\zeta^2/ak}\right)\right]$$

Now obtain cross section:

$$\sigma_{TOT} = \frac{4\pi}{k}\,\mathrm{Im}\, f_k(0) = \frac{4\pi}{k}\,\mathrm{Im}\left\{ik\left[\frac{a^2}{2}+\frac{k^2a^4}{\zeta^4}\left(1+\left(-1+\frac{i\zeta^2}{ak}\right)\left(\cos\frac{\zeta^2}{ak}+i\sin\frac{\zeta^2}{ak}\right)\right)\right]\right\}$$

$$= 4\pi\,\mathrm{Re}\left[\frac{a^2}{2}+\frac{k^2a^4}{\zeta^4}\left(1-\cos\frac{\zeta^2}{ak}-\frac{\zeta^2}{ak}\sin\frac{\zeta^2}{ak}\right)+\text{imaginary}\right]$$

$$= 2\pi a^2\left(1+\frac{2k^2a^2}{\zeta^4}\left(1-\cos\frac{\zeta^2}{ak}-\frac{\zeta^2}{ak}\sin\frac{\zeta^2}{ak}\right)\right)$$