

Irregular spherical Coulomb Wavefunctions $H_l^\pm(\gamma; kr)$ & $G_l(\gamma; kr)$

Defn:

$$H_l^\pm(\gamma; kr) = e^{\pm i[kr - \frac{\pi l}{2} - \gamma \ln(2kr) + \sigma_l(k)]} (\mp 2i kr)^{l+1 \pm i\gamma} \underbrace{U(l+1 \pm i\gamma; 2l+2; \mp 2i kr)}_{\text{Tricomi's confl. Hypergeom. function}}$$

Tricomi's confl. Hypergeom. function

Mathematica: HypergeometricU[$\alpha; \beta; z$]

sometimes called

"Logarithmic Coulomb function".

$$G_l(\gamma; kr) = \frac{1}{2} [H_l^+(\gamma; kr) + H_l^-(\gamma; kr)]$$

Behavior at origin: ($r \rightarrow 0$) (singular at origin)

$$G_l(\gamma; kr) \xrightarrow{r \rightarrow 0} \frac{1}{(2l+1) C_l(\gamma(k))} \left(\frac{1}{kr}\right)^l \left[1 + \begin{cases} O(\gamma kr \ln kr), & l=0 \\ O\left(\frac{\gamma}{r} kr\right), & l \geq 1 \end{cases} \right]$$

Asymptotic behavior at infinity: ($r \rightarrow \infty$)

$$G_l(\gamma; kr) \xrightarrow{\infty} \cos \left[kr - \frac{\pi l}{2} - \gamma(k) \ln(2kr) + \sigma_l(k) \right]$$

Riccati function:

$$u_l(k, r) = e^{i\sigma_l(k)} G_l(\gamma; kr)$$