Analytic Properties of Scattering Amplitudes

Best to use a different form of the Radial Schrödinger equation: remove first derivatives using Riccati's transformation.

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{\ell(\ell+1)}{r^2}\right)+V(r)\right]R_\ell(r) = E\,R_\ell(r)$$

$$\left[\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{\ell(\ell+1)}{r^2}+\underbrace{\frac{2m}{\hbar^2}E}_{\substack{k^2\\ \text{(wavenumber)}}}-\underbrace{\frac{2m}{\hbar^2}V(r)}_{\equiv U(r)}\right]R_\ell(r)=0 \qquad \underline{\text{Reduced potential}}$$

$$\left[\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{\ell(\ell+1)}{r^2}+k^2-U(r)\right]R_\ell(r)=0.$$

Now, set $R_\ell(r)=\frac{1}{r}u_\ell(r)$ to obtain a new diff. eq. for $u_\ell(r)$:

$$\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}-\frac{\ell(\ell+1)}{r^3}+\frac{1}{r}k^2-\frac{1}{r}U(r)\right]u_\ell(r)=0$$

$$\boxed{\left(\frac{\partial^2}{\partial r^2}-\frac{\ell(\ell+1)}{r^2}+k^2-U(r)\right)u_\ell(r)=0}$$

Note: If divided by k^2, can express in terms of dimensionless quantity $\bar{r}=kr$. All k-dependence is inside $\frac{U(r)}{k^2}$.

The free $(U(r)=0)$ solutions are the Riccati-Bessel functions.

	Riccati function	Small r behavior	asymptotic behavior
"sin"	$\hat{j}_\ell(kr)\equiv (kr)\,j_\ell(kr)$	$\frac{2^\ell \ell!}{(2\ell+1)!}(kr)^{\ell+1}+\ldots$	$\simeq \sin\left(kr-\frac{\ell\pi}{2}\right)$
"cos"	$\hat{n}_\ell(kr)\equiv -(kr)\,n_\ell(kr)$	$\frac{(2\ell)!}{2^\ell \ell!}\frac{1}{(kr)^\ell}+\ldots$	$\simeq \cos\left(kr-\frac{\ell\pi}{2}\right)$
"exp"	$\hat{h}^\pm_\ell(kr)\equiv \hat{n}_\ell(kr)\pm i\hat{j}_\ell(kr) = \pm i\,(kr)\,h^{(\frac{1}{2})}_\ell(kr)$	$\frac{(2\ell+1)!}{2^\ell \ell!}\frac{1}{(kr)^\ell}+\ldots$	$\simeq \underbrace{e^{\mp i\frac{\ell\pi}{2}}}_{(i)^{\mp\ell}}e^{\pm ikr}$

- Notice $\hat{h}^\pm$ defined quite analogously to $\exp=\cos\pm i\sin$.
- $\hat{j}_\ell$ is <u>more</u> regular at $r=0$ than $\hat{n}_\ell$ is irregular.

Useful reflection identities:

$$\hat{j}_\ell(-kr)=(-1)^{\ell+1}\hat{j}_\ell(kr) \qquad \hat{j}^*_\ell(z^*)=\hat{j}_\ell(z)$$

$$\hat{n}_\ell(-kr)=(-1)^\ell\,\hat{n}_\ell(kr) \qquad \hat{n}^*_\ell(z^*)=\hat{n}_\ell(z),\quad z\neq 0$$

$$\hat{h}^\pm_\ell(-kr)=(-1)^\ell\,\hat{h}^\mp_\ell(kr) \qquad \left[\hat{h}^\pm_\ell(z^*)\right]^*=\hat{h}^\mp_\ell(z)\quad z\neq 0$$

To make rigorous statements about the analytic properties, make certain assumptions about $U(r)$:

so that centrifugal barrier dominates.

Less singular than $1/r^2$ at origin: $\int_0 dr\, r U(r) \equiv \text{finite}$ $\left(\text{i.e. } U(r) \sim \frac{1}{r^{2-\epsilon}} + \ldots\right)$

Vanishes faster than $1/r^3$ at infinity: $\int^\infty dr\, r^2 U(r) \equiv \text{finite}$

EXAMPLES: Yukawa, Woods-Saxon... but NOT Coulomb ~ does not fall off fast enough.

Potential analytic function of r for $\operatorname{Re} r > 0$ "Analytic potentials"

As before, for each k, there are two linearly independent solutions. Different linear combinations defined by suitable boundary conditions yield solutions with useful analytic or physical properties:

(mixed boundary-value condition)

① PHYSICAL SOLUTIONS: $u_\ell(k,0) = 0$ and $\lim_{r\to\infty} u_\ell(k,r) \to e^{i\delta_\ell(k)} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell(k)\right)$

well-known & studied previously.

② REGULAR SOLUTIONS: $\boxed{\phi_\ell(k,r)}$ (Initial-value condition at $r\to 0$)

$$\lim_{r\to 0} \phi_\ell(k,r) \longrightarrow \hat{j}_\ell(kr) \equiv \frac{2^\ell \ell!}{(2\ell+1)!}(kr)^{\ell+1} + O(r^{\ell+2})$$

— tends to physical, free radial solution at origin.

③ IRREGULAR SOLUTIONS: $\boxed{f_\ell^\pm(k,r)}$ (Initial-value condition at $r\to\infty$)

$$\lim_{r\to\infty} f_\ell^+(k,r) = \hat{h}_\ell^+(kr) \equiv e^{-i\frac{\ell\pi}{2}} e^{ikr}$$

$$\lim_{r\to\infty} f_\ell^-(k,r) = \hat{h}_\ell^-(kr) \equiv e^{i\frac{\ell\pi}{2}} e^{-ikr}$$

— tend to free incoming and outgoing waves at infinity.