Properties of the IRREGULAR SOLUTION, $f_\ell^+(k,r)$ and $f_\ell^-(k,r)$

Satisfies same diff. eq. as $\phi_\ell(k,r) \Rightarrow$ same $G_\ell(r,r')$

But different boundary condition

Integral Equation: $f_\ell^\pm(k,r) = \hat{h}_\ell^\pm(kr) - \int_r^\infty dr'\, G_\ell(r,r')\, U(r')\, f_\ell^\pm(k,r)$

Note: Minus sign, because r is in lower limit $\Rightarrow$ this time, $G_\ell(r,r')=0$ for $r>r'$

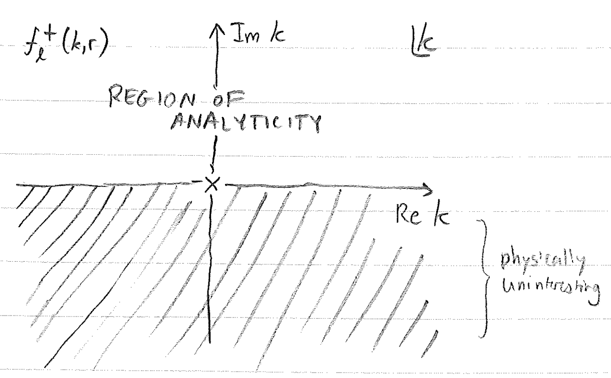
As before, solve by iteration [start with $f_\ell^\pm(k,r) \approx \hat{h}_\ell^\pm(kr)$]

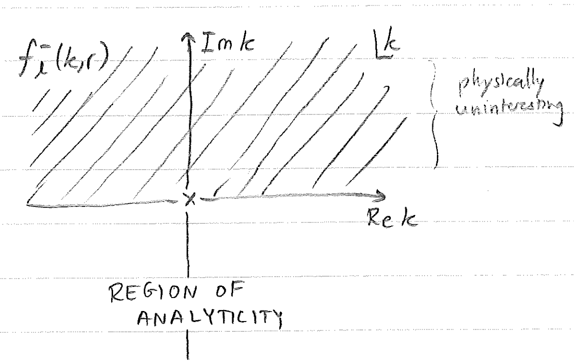
The solution is (reintroducing $U(r) \to \lambda U(r)$):

$$\boxed{\begin{aligned} f_\ell^\pm(k,r) &= \sum_{n=0}^\infty (-1)^n \lambda^n f_\ell^{\pm[n]}(k,r) \\ \text{where } f_\ell^{\pm[n]}(k,r) &= \int_r^\infty dr_n \int_{r_n}^\infty dr_{n-1} \cdots \int_{r_2}^\infty dr_1\, G_\ell(r,r_n) \ldots G_\ell(r_2,r_1)\, U(r_n) \ldots U(r_1)\, \hat{h}_\ell^\pm(kr_1) \\ \text{with } f_\ell^{\pm[0]}(k,r) &= \hat{h}_\ell^\pm(kr) \end{aligned}}$$

BUT, unlike $\phi_\ell(k,r)$, the IRREGULAR SOLUTION $f_\ell^+(k,r)$ is analytic ONLY in the upper-half k-plane ($k>0$), except for a pole of order ℓ at $k=0$.

(Correspondingly, $f_\ell^-(k,r)$ analytic only in lower-half k-plane.)





The integrals for $f_\ell^{\pm[n]}(k,r)$ fail to converge for $\text{Im}\, k < 0$.

$\hat{h}^+(kr)$ grows like $\sim e^{+|\text{Im}\, kr|}$ for large r — see next page.

Existence of integrals and convergence of series.

Follow same steps as before, using $\left|h_\ell^{+}(kr)\right| \le \gamma_\ell \left(\frac{|kr|}{1+|kr|}\right)^{-\ell} e^{-\operatorname{Im} kr}$.

$$\left|G_\ell(r,r')\right| \le \beta_\ell \left| \left(\frac{|kr|}{1+|kr|}\right)^{\ell+1} \left(\frac{|kr'|}{1+|kr'|}\right)^{-\ell} e^{|\operatorname{Im} kr| - |\operatorname{Im} kr'|} - (r \leftrightarrow r') \right|,$$

but this time 2nd term larger for $r < r'$:

$$\le \beta_\ell \left(\frac{|kr|}{1+|kr|}\right)^{-\ell} \left(\frac{|kr'|}{1+|kr'|}\right)^{\ell+1} e^{|\operatorname{Im} k| \overbrace{(r'-r)}^{\text{positive}}}$$

Now place bound on each term in sum:

$$\left|f_\ell^{\pm[n]}(k,r)\right| \le \int_r^\infty dr_n \cdots \int_{r_2}^\infty dr_1 \left| \beta_\ell \left(\frac{|kr|}{1+|kr|}\right)^{-\ell} \left(\frac{|kr_n|}{1+|kr_n|}\right)^{\cancel{\ell+1}} e^{|\operatorname{Im} k|(\cancel{r_n}-r)} \right.$$

$$\left. \cdots \beta_\ell \cancel{\left(\frac{|kr_2|}{1+|kr_2|}\right)^{-\ell}} \left(\frac{|kr_1|}{1+|kr_1|}\right)^{\ell+1} e^{|\operatorname{Im} k|(r_1-\cancel{r_2})} U(r_n)\cdots U(r_1)\, \gamma_\ell \cancel{\left(\frac{|kr_1|}{1+|kr_1|}\right)^{-\ell}} e^{-\operatorname{Im} k\, r_1} \right|$$

If $\operatorname{Im} k > 0$, these cancel, and convergence follows.

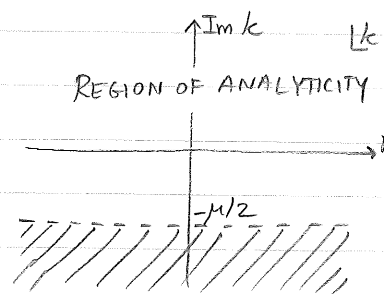
If $\operatorname{Im} k < 0$, we have:

$$\left|f_\ell^{\pm[n]}(k,r)\right| \le \gamma_\ell e^{|\operatorname{Im} kr|} \underbrace{\left(\frac{|kr|}{1+|kr|}\right)^{-\ell}}_{\text{POLE at } k=0 \text{ of order } \ell.} \int_r^\infty dr_n\, \beta_\ell \frac{|kr_n|}{1+|kr_n|} |U(r_n)| \cdots \underbrace{\int_{r_2}^\infty dr_1\, \beta_\ell \frac{|kr_1|}{1+|kr_1|} |U(r_1)|\, e^{2|\operatorname{Im} k| r_1}}$$

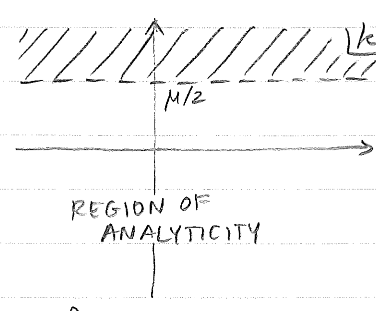
The r_1 integral fails to converge.

Region of analyticity is larger if potential vanishes exponentially quickly so that $\int^\infty dr\, e^{\mu r} |U(r)| \equiv$ convergent. $(\mu > 0)$

Then, provided $\operatorname{Im} k > -\frac{\mu}{2}$, we have $\int_{r_2}^\infty dr_1\, \beta_\ell \frac{|kr_1|}{1+|kr_1|} \overbrace{|U(r_1)|\, e^{\mu r_1}} =$ convergent.



for $f_\ell^{+}(k,r)$



for $f_\ell^{-}(k,r)$

Furthermore, if $U(r)$ is truncated at some $r = r_t$ (vanishes identically for $r > r_t$), then the integrations necessarily converges for all k $\Rightarrow$ IRREGULAR SOLUTIONS, $f_\ell^\pm(k,r)$, are entire for truncated potentials.

— Turns out that behavior of $f_\ell^+(k,r)$ in lower-half plane is very sensitive to the tail of the potential — not very interesting, physically.

Other properties of the IRREGULAR SOLUTION, $f_\ell^\pm(k,r)$

Using the solution to integral equation, we have
ONLY IN THE REGION OF ANALYTICITY / CONVERGENCE,

$$\left[f_\ell^{\pm[n]}(k^*,r)\right]^* = f_\ell^{\mp[n]}(k,r) \quad \Rightarrow \quad \boxed{\left[f_\ell^{\pm}(k^*,r)\right]^* = f_\ell^{\mp}(k,r)}$$

AND

$$f_\ell^{\pm[n]}(-k,r) = (-1)^\ell f_\ell^{\mp[n]}(k,r) \quad \Rightarrow \quad \boxed{f_\ell^{\pm}(-k,r) = (-1)^\ell f_\ell^{\mp}(k,r)}$$