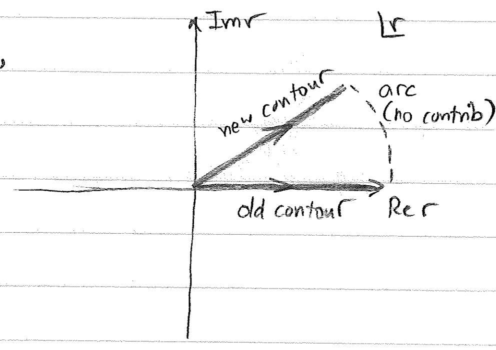
Analyticity of $S_\ell(k)$: (for typical potentials, not exponentially vanishing at $r\to\infty$)

$\left.\begin{array}{l} f_\ell(k) \text{ analytic in upper-half plane} \\ f_\ell(-k) \text{ analytic in lower-half plane} \end{array}\right\} \Rightarrow$ no region of analyticity for $S(k) = \dfrac{f_\ell(-k)}{f_\ell(k)}$?

Can do better: can establish analyticity in the lower half plane:

Since $\phi_\ell(k,r)$ (and $\hat{h}^+_\ell(kr)$ & $U(r)$) analytic for $r>0$, can rotate contour integral (arc contribution vanishes)



$$f_\ell(k) = 1 + \frac{1}{k}\int_0^r dr\ \hat{h}^+_\ell(kr)\,U(r)\,\phi_\ell(k,r)$$

$$= 1 + \frac{1}{k}\int_0^{\infty e^{i\theta}} dr\ \hat{h}^+_\ell(kr)\,U(r)\,\phi_\ell(k,r)$$

ch. integration variables: $r \to re^{-i\theta}$

$$= 1 + \frac{e^{-i\theta}}{k}\int_0^\infty dr\ \hat{h}^+_\ell(kre^{-i\theta})\,U(re^{-i\theta})\,\phi_\ell(k,re^{-i\theta})$$

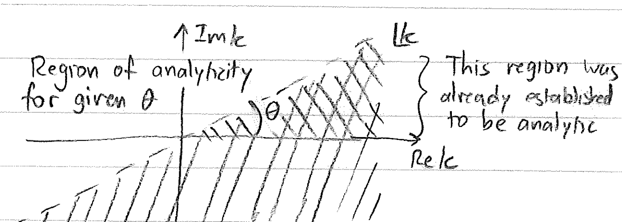
Then, bounds become:

$$|f_\ell(k,r)-1| \le \text{const.}\left|\frac{1}{k}\right|\int_0^\infty dr\left(\frac{|kre^{-i\theta}|}{1+|kre^{-i\theta}|}\right)|U(re^{-i\theta})|\,e^{-\operatorname{Im}(kre^{-i\theta})+|\operatorname{Im}(kre^{-i\theta})|}$$

$\operatorname{Im}(kre^{-i\theta}) = -r\operatorname{Im}(ke^{-i\theta})$

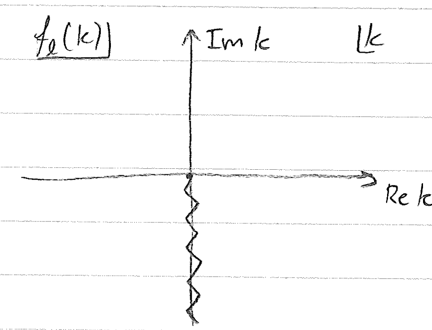
So, integral converges if $\operatorname{Im}(ke^{-i\theta})>0$

$\Rightarrow f_\ell(k)$ analytic for $\operatorname{Im}(ke^{-i\theta})>0$.

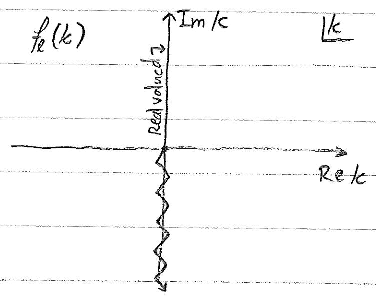


And since $\phi_\ell, \hat{h}^+_\ell$ & $U(r)$ analytic for θ satisfying $-\pi/2 < \theta < \pi/2$, union of regions covered by $\Rightarrow$

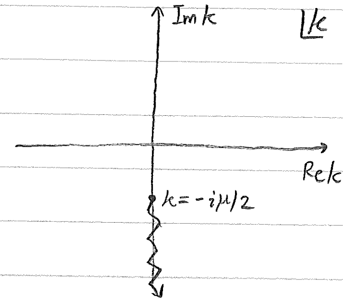
$\boxed{f_\ell(k) \text{ analytic for all } k\text{, except for a cut down the negative imaginary axis.}}$



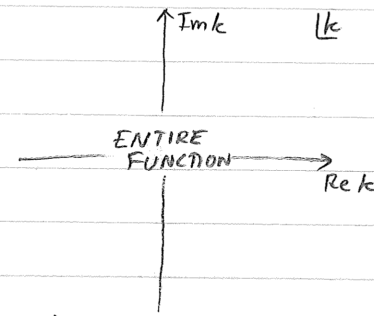
Analyticity of Jost function for various potentials:



for $U(r\to\infty) \sim \frac{1}{r^{2-\epsilon}}$

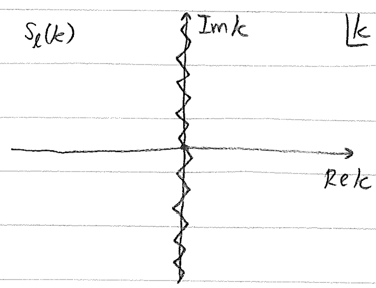


for $U(r\to\infty) \sim e^{-\mu r}$

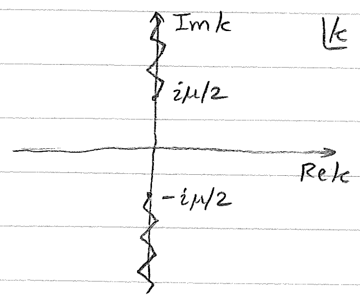


for $U(r\to\infty)=0$
(truncated potentials)

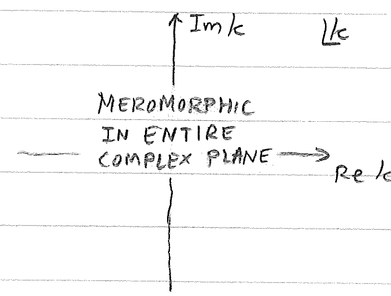
So S-matrix elements, $S_\ell(k) = \frac{f_\ell(-k)}{f_\ell(k)}$ are meromorphic — analytic except for the existence of isolated poles, due to zeros of $f_\ell(k)$.



for $U(r\to\infty) \sim \frac{1}{r^{2-\epsilon}}$



for $U(r\to\infty) \sim e^{-\mu r}$



for $U(r\to\infty) = 0$