Bound States and Poles of the S-Matrix

Recall: $\phi_\ell(k,r) = \frac{i}{2}\left[f_\ell(k)\, f_\ell^-(k,r) - f_\ell(-k)\, f_\ell^+(k,r)\right]$

In some points, say e.g. $\bar{k}$, in the complex k-plane, $f_\ell(k)$ may vanish: $f_\ell(\bar{k})=0$.

Cannot vanish on real axis because: for $k \in \mathbb{R}$, $[f_\ell(k)]^* = f_\ell(-k)$,

so if $f_\ell(\bar{k}) = 0 \Rightarrow [f_\ell(\bar{k})]^* = f_\ell(-\bar{k}) = 0$.

$\therefore$ $\phi_\ell(k,r) = 0$, which does not satisfy defining boundary conditions for ϕ_ℓ.

If $f_\ell(k)$ vanishes on (positive) imaginary axis: $\bar{k} \equiv i\,\mathrm{Im}\,\bar{k}$,

$$\phi_\ell(\bar{k},r) = \frac{i}{2} \times -f_\ell(-\bar{k})\, f_\ell^+(\bar{k},r), \text{ which for large } r$$

$$\xrightarrow{r\to\infty} \frac{-i}{2} f_\ell(-\bar{k})\, e^{-i\frac{\ell\pi}{2}} e^{ikr}$$

$$\equiv -\frac{i}{2} f_\ell(-\bar{k})\, e^{-i\frac{\ell\pi}{2}} e^{-(\mathrm{Im}\,k)r}$$

is a normalizable solution to the radial Schrödinger equation,

with energy $E = \frac{k^2}{2m}\hbar^2 \equiv -\frac{(\mathrm{Im}\,k)^2}{2m}\hbar^2$ and ang. mom $=\ell$.

$\Rightarrow$ corresponds to a <u>bound state</u> of $\hat{H}$.

Conversely, if $\hat{H}$ has a bound state with (negative) energy $E = \frac{\bar{k}^2}{2m}\hbar^2 < 0$, and angular momentum $=\ell$, the wavefunction must be a normalizable solution with $\bar{k}$ pure positive imaginary $\equiv i\,\mathrm{Im}\,\bar{k}$.

Since $f_\ell^\pm(\bar{k},r) \xrightarrow{r\to\infty} e^{\mp(\mathrm{Im}\,\bar{k})r}$, wavefunction $\phi_\ell(\bar{k},r)$ may not contain any $f_\ell^-(\bar{k},r) \Rightarrow \underline{f_\ell(\bar{k},r) = 0}$.

In summary,

$f_\ell(\bar{k}) = 0$, for $\bar{k}$ positive imaginary	$\Leftrightarrow$	Bound State of $\hat{H}$ with energy $E = -\frac{(\mathrm{Im}\,\bar{k})^2}{2m}\hbar^2$

Furthermore $f_\ell(\bar{k})$ is a <u>simple zero</u> $\Rightarrow S_\ell(k) = \dfrac{f_\ell(-k)}{f_\ell(k)}$ contains a simple pole at $\bar{k}$, and corresponds to a bound state.

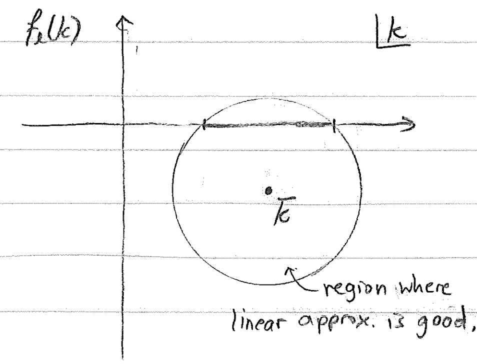
Resonances

Idea: $\boxed{\text{Poles of } S_\ell(k) \text{ in lower-half } k\text{-plane}} \overset{\text{sometimes}}{\Longleftrightarrow} \boxed{\text{Resonance with ang. mom. } \ell}$

CAREFUL: $S_\ell(k)$ is non-analytic in lower-half k-plane — can't make any statements unless $U(r\to\infty)\to 0$ quickly enough to permit analytic continuation.

Let $\bar{k}$ be a zero of $f_\ell(k)$ in lower half k-plane:

$$f_\ell(\bar{k}) = 0$$



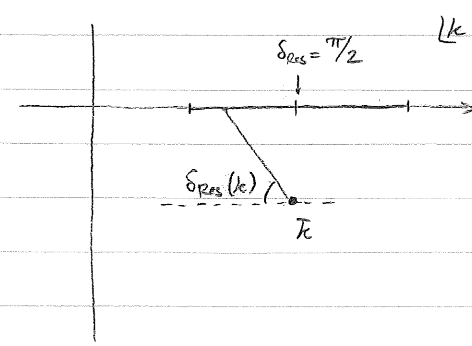
Investigate behavior of $f_\ell(k)$ in the neighborhood of $\bar{k}$ (remember, $f_\ell(k)$ assumed analytic at $\bar{k}$)

$$f_\ell(k) \approx \underset{\overset{\shortparallel}{0}}{f_\ell(\bar{k})} + (k-\bar{k})\left.\frac{df_\ell}{dk}\right|_{\bar{k}} + \mathcal{O}(k-\bar{k})^2$$

If $\bar{k}$ is close to Real-k axis, there is a domain (shown) on the physical region where linear approx. is good. The phase shift, $\delta_\ell(k) = -\arg f_\ell(k)$, on the domain is:

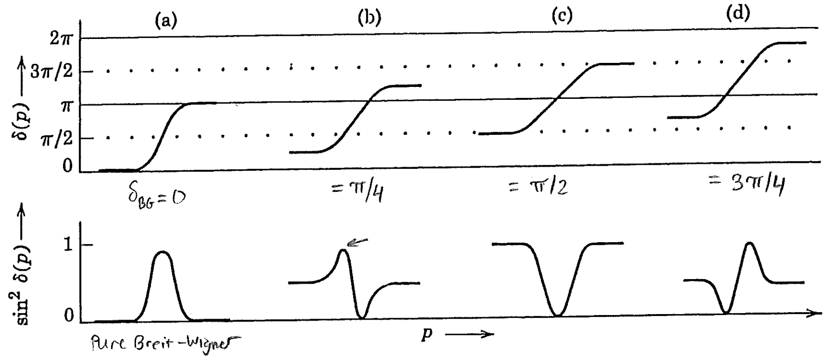
$$\delta_\ell(k) \approx -\arg\left[(k-\bar{k})\left.\frac{df_\ell}{dk}\right|_{\bar{k}}\right]$$

$$= \underbrace{-\arg\left(\frac{df_\ell}{dk}\right)_{\bar{k}}}_{\delta_{BG}(k)} + \underbrace{-\arg(k-\bar{k})}_{\delta_{Res}(k)}$$



Suppose $\delta_{BG}(k) \approx$ const. As k increases passed the zero at $\bar{k}$, resonant phase shift goes from $0 \to \pi$. THE CLOSER $\bar{k}$ is to physical region (real k-axis), the more sudden the change of $\delta_{Res}(k)$ is.

Depending on the value of $\delta_{BG}(k)$, there are four qualitatively different kinds of resonances — see next page.



$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l(k)$

FIGURE 13.3. **Four possible resonances. The $\delta(p)$ plots show the resonant phase shifts for $\delta_{bg} = 0$, $\pi/4$, $\pi/2$, and $3\pi/4$. The $\sin^2 \delta(p)$ plots show the corresponding behavior of the partial cross section [apart from a factor $4\pi(2l + 1)/p^2$].**

— A small positive background phase shift advances the total phase shift, so that the resonance peak appears "earlier" (at a smaller k)