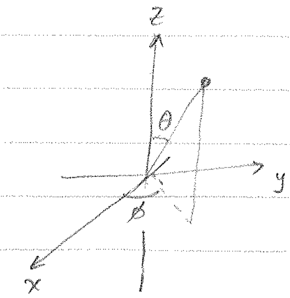
$$\begin{pmatrix} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{pmatrix}$$

<u>Orbital angular momentum in polar coordinates.</u>



$$\vec{\nabla} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z}$$

$$= \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Using $\hat{\vec{p}} = -i\hbar\vec{\nabla}$ and $\vec{x} = \vec{e}_r r$;

$$\hat{\vec{L}} = \vec{x}\times\vec{p} = -i\hbar\left(\vec{x}\times\vec{\nabla}\right)$$

$$\boxed{\begin{aligned} \vec{L}_x &= -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right) \\ \vec{L}_y &= -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right) \\ \vec{L}_z &= -i\hbar\frac{\partial}{\partial\phi} \end{aligned}}$$

$$\boxed{\begin{aligned} \hat{L}_\pm &= \hat{L}_x \pm i\hat{L}_y = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) \\ \vec{L}^2 &= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \end{aligned}}$$

inverses.

$$\begin{Bmatrix} L_x = \frac{1}{2}(L_+ + L_-) \\ L_y = \frac{1}{2i}(L_+ - L_-) \end{Bmatrix}$$

<u>The eigenvalue equation:</u>

$$\hat{\vec{L}}^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle$$

$$L_z|l,m\rangle = \hbar m|l,m\rangle$$

$$\boxed{\begin{aligned} -\cancel{\hbar^2}\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]Y_l^m(\theta,\phi) &= \cancel{\hbar^2}\, l(l+1)\,Y_l^m(\theta,\phi) \\ -i\cancel{\hbar}\frac{\partial}{\partial\phi}Y_l^m(\theta,\phi) &= \cancel{\hbar}\, m\, Y_l^m(\theta,\phi) \end{aligned}}$$