Spherical harmonics - summary

Eigenvalue equations:

① $$\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] Y_\ell^m(\theta,\phi) = -\ell(\ell+1)\, Y_\ell^m(\theta,\phi)$$

② $$\frac{\partial}{\partial\phi} Y_\ell^m(\theta,\phi) = im\, Y_\ell^m(\theta,\phi)$$

Separation ansatz: $Y_\ell^m(\theta,\phi) = f_\ell^m(\theta)\, g_\ell^m(\phi)$

$\boxed{\text{Equation ②}}$ $$\Rightarrow \cancel{f_\ell^m(\theta)}\frac{\partial}{\partial\phi} g_\ell^m(\phi) = im\, \cancel{f_\ell^m(\theta)}\, g_\ell^m(\phi)$$

$$\Rightarrow g_\ell^m(\phi) = e^{im\phi} \qquad \text{indep. of } \ell$$

boundary condition: $g_\ell^m(\phi) = g_\ell^m(\phi+2\pi) \Rightarrow m \in \mathbb{Z}$ (integer)

$\boxed{\text{Equation ①}}$

$$\left[\frac{-m^2}{\sin^2\theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] f_\ell^m(\theta)\, \cancel{g_\ell^m(\phi)} = -\ell(\ell+1)\, f_\ell^m(\theta)\, \cancel{g_\ell^m(\phi)}$$

ODE $$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \frac{m^2}{\sin^2\theta} + \ell(\ell+1)\right] f_\ell^m(\theta) = 0$$

Put: $\cos\theta = z \Rightarrow \sin\theta = \sqrt{1-z^2}$ and $\frac{d}{d\theta} = \sqrt{1-z^2}\,\frac{d}{dz}$

so that

$$\left[\frac{1}{\sqrt{1-z^2}}\left(\sqrt{1-z^2}\frac{d}{dz}\right)\left[\sqrt{1-z^2}\left(\sqrt{1-z^2}\frac{d}{dz}\right)\right] - \frac{m^2}{1-z^2} + \ell(\ell+1)\right] f_\ell^m(z) = 0$$

$$\left[(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} - \frac{m^2}{1-z^2} + \ell(\ell+1)\right] f_\ell^m(z) = 0.$$

(well-known)
solution: $f_\ell^m(z) = P_\ell^m(z)$ (associated Legendre polynomials).

Summary

In Mathematica

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi}$$

SphericalHarmonicY[ℓ, m, θ, φ]

$m>0$
$$P_\ell^m(z) = (-1)^m (1-z^2)^{m/2} \frac{d^m}{dz^m} P_\ell(z)$$

LegendreP[ℓ, m, z]

$m<0$
$$P_\ell^{-|m|}(z) = \underbrace{(-1)^{|m|}}_{\text{Condon-Shortley phase}} \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^{|m|}(z)$$

$$P_\ell(z) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dz^\ell}(z^2-1)^\ell$$

(Rodrigues' formula) LegendreP[ℓ, z]

Both cases:

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}} \times \left\{\begin{matrix} 1 & \text{if } m>0 \\ (-1)^{|m|} & \text{if } m<0 \end{matrix}\right\} P_\ell^{|m|}(\cos\theta)\, e^{im\phi}$$

Orthogonality:

$$\int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi\; Y_\ell^m(\theta,\phi)\, Y_{\ell'}^{*\,m'}(\theta,\phi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

Completeness:

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\phi)\, Y^{*\,m}_\ell(\theta',\phi') = \underbrace{\frac{1}{\sin\theta}\,\delta(\theta-\theta')\,\delta(\phi-\phi')}$$

$$= \delta(\Omega-\Omega') \quad \text{(shorthand)}$$

Addition theorem:

$$\sum_{m=-\ell}^{\ell} Y^{*\,m}_\ell\underbrace{(\theta',\phi')}\, Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\, Y_\ell^0(\Theta, 0) = \frac{2\ell+1}{4\pi} P_\ell(\cos\Theta)$$

direction angle for vector 1 — and vector 2

(with respect to some arbitrary axis)

angle between two vectors (see next page)

can be any value (doesn't matter)

Useful reminder: angles inside $Y_\ell^m(\Omega)$ are measured against the axis of angular momentum quantization described by the (ℓ, m) numbers.