Regge Poles

If $E<0$, poles of $f_\ell(E)$ (scattering amplitude) or zeros of $f_\ell(k)$ (Jost function) must occur at real values of $\ell = \alpha(E)$, for $\mathrm{Re}\,\alpha > 1/2$.

PROOF

If $f_\ell(k)=0$, the regular function $\phi_\ell(k,r)$ is regular at both $r=0$ & $r\to\infty$.

Start with the radial eqn & its complex conjugate:

$$\phi^*\left[\frac{\partial^2}{\partial r^2} - \frac{\alpha(\alpha+1)}{r^2} + k^2 - \cancel{U(r)}\right]\phi - \phi\left[\frac{\partial^2}{\partial r^2} - \frac{\alpha^*(\alpha^*+1)}{r^2} + (k^*)^2 - \cancel{U(r)}\right]\phi^* = 0$$

$\phi^* \equiv \phi^*_{\ell^*}(k^*, r)$

$$\underbrace{\phi^*\frac{\partial^2}{\partial r^2}\phi - \phi\frac{\partial^2}{\partial r^2}\phi^*} - \frac{2\,\mathrm{Im}(\alpha(\alpha+1))}{r^2}\phi^*\phi + \underbrace{\left(k^2-(k^*)^2\right)}\phi^*\phi = 0$$

$E<0 \Rightarrow k$ purely imaginary $\Rightarrow (k^*)^2 = k^2$

$$\frac{\partial}{\partial r}\left(\phi^*\frac{\partial\phi}{\partial r} - \phi\frac{\partial\phi^*}{\partial r}\right) - 2\,\mathrm{Im}\,\alpha(\alpha+1)\frac{|\phi|^2}{r^2} = 0.$$

Integrate over r: $0\to\infty$, First term vanishes (total derivative), and surface term vanishes since $\phi_\ell(r\to\infty)\to 0$ (bound state).

$$\mathrm{Im}\,\alpha(\alpha+1)\int_0^\infty dr\,\frac{|\phi_\ell(k,r)|^2}{r^2} = 0.$$

If the integral converges at origin (true if $\mathrm{Re}\,\alpha > 1/2$), then

$$\mathrm{Im}\,\alpha(\alpha+1) = 0$$

Write $\alpha(E) = a + bi$. $\quad$ ($a \equiv \mathrm{Re}\,\alpha(E)$, $b \equiv \mathrm{Im}\,\alpha(E)$)

Then, $\mathrm{Im}\,(a+bi)(a+bi+1) = 0$

$\mathrm{Im}\,(a^2 + abi + a + bia - b^2 + bi) = 0$

$2ab + b = 0$

$b(2a+1) = 0 \Rightarrow \boxed{b \equiv \mathrm{Im}\,\alpha(E) = 0}$ or $\mathrm{Re}\,\alpha(E) = -1/2$

QED

this solution irrelevant since $\mathrm{Re}\,\alpha > 1/2$