1 Complex Conjugation Identities

$$
\begin{aligned}
\hat{\jmath}_\ell(z)^* &= \hat{\jmath}_{\ell^*}(z^*)\\
\hat{n}_\ell(z)^* &= \hat{n}_{\ell^*}(z^*)\\
\hat{h}^+_\ell(z)^* &= \hat{h}^-_{\ell^*}(z^*)\\
\hat{h}^-_\ell(z)^* &= \hat{h}^+_{\ell^*}(z^*)
\end{aligned}
$$

2 Reflection Formulas $z \to -z$

$$
\begin{aligned}
\hat{\jmath}_\ell(z\,e^{\pm i\pi}) &= -e^{\pm i\pi\ell}\hat{\jmath}_\ell(z)\\
\hat{n}_\ell(z\,e^{\pm i\pi}) &= e^{\mp i\pi\ell}\hat{n}_\ell(z) - 2\sin(\pi\ell)\hat{\jmath}_\ell(z)\\
\hat{h}^\pm_\ell(z\,e^{\pm i\pi}) &= e^{\mp i\pi\ell}\hat{h}^\mp_\ell(z)\\
\hat{h}^\mp_\ell(z\,e^{\pm i\pi}) &= e^{\mp i\pi\ell}\hat{h}^\pm_\ell(z) - 4\sin(\pi\ell)\hat{\jmath}_\ell(z)
\end{aligned}
$$

3 Circuit Formulas

$$
\begin{aligned}
\hat{\jmath}_\ell(z\,e^{\pm 2\pi i}) &= e^{\pm 2\pi i\ell}\hat{\jmath}_\ell(z)\\
&= \hat{\jmath}_\ell(z) \pm 2ie^{\pm i\pi\ell}\sin(\pi\ell)\hat{\jmath}_\ell(z)\\
\hat{n}_\ell(z\,e^{\pm 2\pi i}) &= e^{\mp 2\pi i\ell}\hat{n}_\ell(z) \pm 4i\sin^2(\pi\ell)\hat{\jmath}_\ell(z)\\
\hat{h}^+_\ell(z\,e^{2\pi i}) &= \hat{h}^+_\ell(z) - 2i\sin(\pi\ell)\hat{h}^+_\ell(z\,e^{i\pi})\\
\hat{h}^-_\ell(z\,e^{-2\pi i}) &= \hat{h}^-_\ell(z) + 2i\sin(\pi\ell)\hat{h}^-_\ell(z\,e^{-i\pi})
\end{aligned}
$$

4 Reflection Formulas $\ell \to -\ell - 1$

$$
\begin{aligned}
\hat{\jmath}_{-\ell-1}(z) &= \cos(\pi\ell)\hat{n}_\ell(z) - \sin(\pi\ell)\hat{\jmath}_\ell(z)\\
&= \cos(\pi\ell)\hat{h}^+_\ell(z) - ie^{-i\pi\ell}\hat{\jmath}_\ell(z)\\
&= \cos(\pi\ell)\hat{h}^-_\ell(z) + ie^{i\pi\ell}\hat{\jmath}_\ell(z)
\end{aligned}
$$

$$
\begin{aligned}
\hat{n}_{-\ell-1}(z) &= -\cos(\pi\ell)\hat{\jmath}_\ell(z) - \sin(\pi\ell)\hat{n}_\ell(z) \\
&= i\cos(\pi\ell)\hat{h}^+_\ell(z) - ie^{-i\pi\ell}\hat{n}_\ell(z) \\
&= -i\cos(\pi\ell)\hat{h}^-_\ell(z) + ie^{i\pi\ell}\hat{n}_\ell(z)
\end{aligned}
$$

$$
\begin{aligned}
\hat{h}^+_{-\ell-1}(z) &= ie^{i\pi\ell}\hat{h}^+_\ell(z) \\
\hat{h}^-_{-\ell-1}(z) &= -ie^{-i\pi\ell}\hat{h}^-_\ell(z)
\end{aligned}
$$

5 Behavior near origin $z \to 0$

$$
\hat{\jmath}_\ell(z) \to \begin{cases} \dfrac{2^\ell\Gamma(\ell+1)}{\Gamma(2\ell+2)} z^{\ell+1} + \mathcal{O}(z^{\ell+3}) & \mathrm{Re}(\ell) > -1/2 \\[2ex] -\cos(\pi\ell)\dfrac{2\Gamma(-2\ell-1)}{2^{-\ell}\Gamma(-\ell)} z^{\ell+1} + \mathcal{O}(z^{\ell+3}) & \mathrm{Re}(\ell) < -1/2 \end{cases}
$$

$$
\hat{n}_\ell(z) \to \begin{cases} \dfrac{\Gamma(2\ell+1)}{2^\ell\Gamma(\ell+1)} z^{-\ell} + \mathcal{O}(z^{-\ell+2}) & \mathrm{Re}(\ell) > -1/2 \\[2ex] -\sin(\pi\ell)\dfrac{2\Gamma(-2\ell-1)}{2^{-\ell}\Gamma(-\ell)} z^{\ell+1} + \mathcal{O}(z^{\ell+3}) & \mathrm{Re}(\ell) < -1/2 \end{cases}
$$