Symmetric tensor identities

Contraction of $p_{k\mu_1}$ into a symmetric tensor. $\{[p_1]^{n_1}[p_2]^{n_2}[g]^r\}^{\mu_1\cdots\mu_p}$

Extract μ_1 index:

$$\{[p_1]^{n_1}[p_2]^{n_2}[g]^r\}^{\mu_1\cdots\mu_p} = p_1^{\mu_1}\{[p_1]^{n_1-1}[p_2]^{n_2}[g]^r\}^{\mu_2\cdots\mu_p}$$

$$+ p_2^{\mu_1}\{[p_1]^{n_1}[p_2]^{n_2-1}[g]^r\}^{\mu_2\cdots\mu_p} + \sum_{i=2}^{p} g^{\mu_1\mu_i}\{[p_1]^{n_1}[p_2]^{n_2}[g]^{r-1}\}^{\mu_2\cdots\hat{\mu}_i\cdots\mu_p}$$

Contract:

$$(p_k)_{\mu_1}\{[p_1]^{n_1}[p_2]^{n_2}[g]^r\}^{\mu_1\cdots\mu_p}$$

$$= p_k\cdot p_1\{[p_1]^{n_1-1}[p_2]^{n_2}[g]^r\}^{\mu_2\cdots\mu_p} + p_k\cdot p_2\{[p_1]^{n_1}[p_2]^{n_2-1}[g]^r\}^{\mu_2\cdots\mu_p}$$

$$+ \sum_{i=2}^{p} p_k^{\mu_i}\{[p_1]^{n_1}[p_2]^{n_2}[g]^{r-1}\}^{\mu_2\cdots\hat{\mu}_i\cdots\mu_p}$$

$$= \sum_{l=1}^{2} p_k\cdot p_l\{[\widehat{p_l}][p_1]^{n_1}[p_2]^{n_2}[g]^r\}^{\mu_2\cdots\mu_p} + (n_k+1)\{[p_k][p_1]^{n_1}[p_2]^{n_2}[g]^{r-1}\}^{\mu_2\cdots\mu_p}$$

More generally,

$$= \sum_{l=1}^{N} p_k\cdot p_l\left\{[\hat{p}_l][p_1]^{n_1}\cdots[p_N]^{n_N}[g]^r\right\}^{\mu_2\cdots\mu_p} + (n_k+1)\left\{[p_k][p_1]^{n_1}\cdots[p_N]^{n_N}[g]^{r-1}\right\}^{\mu_2\cdots}$$

To contract $g_{\mu_1\mu_2}$ into symmetric tensor, need to isolate μ_1 and μ_2.

def: $\bar{\delta}_{ij} = (1-\delta_{ij}) = \begin{cases} 0, & i=j \\ 1, & i\neq j \end{cases}$

$$\{[p_1]^{n_1}[p_2]^{n_2}[g]^r\}^{\mu_1\cdots\mu_p}$$

- Isolate μ_1

$$= \bar{\delta}_{n_1 0}\, p_1^{\mu_1}\{[p_1]^{n_1-1}[p_2]^{n_2}[g]^r\}^{\mu_2\cdots\mu_p} + \bar{\delta}_{n_2 0}\, p_2^{\mu_1}\{[p_1]^{n_1}[p_2]^{n_2-1}[g]^r\}^{\mu_2\cdots\mu_p}$$
$$+ \sum_{i=2}^{p} g^{\mu_1\mu_i}\{[p_1]^{n_1}[p_2]^{n_2}[g]^{r-1}\}^{\mu_2\cdots\hat{\mu}_i\cdots\mu_p}.$$

- Isolate μ_2

$$= \bar{\delta}_{n_1 0}\bar{\delta}_{n_1 1}\, p_1^{\mu_1} p_1^{\mu_2}\{[p_1]^{n_1-2}[p_2]^{n_2}[g]^r\}^{\mu_3\cdots\mu_p} \quad ✓$$
$$+ \bar{\delta}_{n_1 0}\bar{\delta}_{n_2 0}\, p_1^{\mu_1} p_2^{\mu_2}\{[p_1]^{n_1-1}[p_2]^{n_2-1}[g]^r\}^{\mu_3\cdots\mu_p} \quad ✓$$
$$+ \bar{\delta}_{n_1 0}\sum_{i=3}^{p} p_1^{\mu_1} g^{\mu_2\mu_i}\{[p_1]^{n_1-1}[p_2]^{n_2}[g]^{r-1}\}^{\mu_3\cdots\hat{\mu}_i\cdots\mu_p}$$

$$+ \bar{\delta}_{n_2 0}\bar{\delta}_{n_1 0}\, p_2^{\mu_1} p_1^{\mu_2}\{[p_1]^{n_1-1}[p_2]^{n_2-1}[g]^r\}^{\mu_3\cdots\mu_p} \quad ✓$$
$$+ \bar{\delta}_{n_2 0}\bar{\delta}_{n_2 1}\, p_2^{\mu_1} p_2^{\mu_2}\{[p_1]^{n_1}[p_2]^{n_2-2}[g]^r\}^{\mu_3\cdots\mu_p} \quad ✓$$
$$+ \bar{\delta}_{n_2 0}\sum_{i=3}^{p} p_2^{\mu_1} g^{\mu_2\mu_i}\{[p_1]^{n_1}[p_2]^{n_2-1}[g]^{r-1}\}^{\mu_3\cdots\hat{\mu}_i\cdots\mu_p}$$

suppose $i\neq 2$

$$+ \sum_{i=3}^{p} g^{\mu_1\mu_i}\bar{\delta}_{n_1 0}\, p_1^{\mu_2}\{[p_1]^{n_1-1}[p_2]^{n_2}[g]^{r-1}\}^{\mu_3\cdots\hat{\mu}_i\cdots\mu_p}$$
$$+ \sum_{i=3}^{p} g^{\mu_1\mu_i}\bar{\delta}_{n_2 0}\, p_2^{\mu_2}\{[p_1]^{n_1}[p_2]^{n_2-1}[g]^{r-1}\}^{\mu_3\cdots\hat{\mu}_i\cdots\mu_p}$$
$$+ \sum_{i=3}^{p} g^{\mu_1\mu_i}\sum_{\substack{j=4 \\ (j\neq i)}}^{p} g^{\mu_2\mu_j}\{[p_1]^{n_1}[p_2]^{n_2}[g]^{r-2}\}^{\mu_3\cdots\hat{\mu}_i\cdots\hat{\mu}_j\cdots\mu_p}$$

if $i=2$, nothing to isolate

$$+ g^{\mu_1\mu_2}\{[p_1]^{n_1}[p_2]^{n_2}[g]^{r-1}\}^{\mu_3\cdots\mu_p}$$

Proceed to contract with $g_{\mu_1\mu_2}$

$$g_{\mu_1\mu_2}\{[p_1]^{n_1}[p_2]^{n_2}[g]^r\}^{\mu_1\cdots\mu_p}$$

$$= \sum_{i,j=1}^{2} p_i\cdot p_j \,\{[\widehat{p_i}][\widehat{p_j}]\,[p_1]^{n_1}[p_2]^{n_2}[g]^r\}^{\mu_3\cdots\mu_p} \quad \leftarrow \text{lines } 1,2,4,5.$$

redundant

$$+2n_1\,\bar{\delta}_{r,0}\,\bar{\delta}_{n_1 0}\,\{[p_1]^{n_1}[p_2]^{n_2}[g]^{r-1}\}^{\mu_3\cdots\mu_p} \quad \leftarrow \text{line } 3,7$$

$$+2n_2\,\bar{\delta}_{r,0}\,\bar{\delta}_{n_2 0}\,\{[p_1]^{n_1}[p_2]^{n_2}[g]^{r-1}\}^{\mu_3\cdots\mu_p} \quad \leftarrow \text{line } 6,8$$

$$+\text{Bool}(r\geq 2)\sum_{i=3}^{p}\sum_{\substack{j=4\\(j\neq i)}}^{p} \underbrace{g_{\mu_1\mu_2}\,g^{\mu_1\mu_i}\,g^{\mu_2\mu_j}}_{g^{\mu_i\mu_j}\cdots}\,\{[p_1]^{n_1}[p_2]^{n_2}[g]^{r-2}\}^{\mu_3\cdots\hat{\mu}_i\cdots\hat{\mu}_j\cdots\mu_p} \quad \leftarrow \text{sum } g \text{ lines } (2r-1)\times\{\ \}^{\mu_3\cdots\mu_p}$$

$$+\bar{\delta}_{r,0}\,d\,\{[p_1]^{n_1}[p_2]^{n_2}[g]^{r-1}\}^{\mu_3\cdots\mu_p}.$$

$$= \sum_{i,j} p_i\cdot p_j\,\{[\widehat{p_i}][\widehat{p_j}]\,[p_1]^{n_1}[p_2]^{n_2}[g]^r\}^{\mu_3\cdots\mu_p}$$

$$+\underbrace{\Big(\overset{\bar{\delta}_{r,0}}{d}+\overbrace{2r-2}^{r\geq 2}+\overbrace{2n_1+2n_2}^{\bar{\delta}_{r,0}}\Big)}_{d+\underbrace{2r+n_1+n_2}_{p}-2+n_1+n_2}\{[p_1]^{n_1}[p_2]^{n_2}[g]^{r-1}\}^{\mu_3\cdots\mu_p}$$

sum over unique vectors.
NOT over every repetition of vectors.

$$r=\tfrac{1}{2}(p-n_1-n_2-\ldots)$$

$$= \sum_{i,j} p_i\cdot p_j\,\{[\hat{p}_i][\hat{p}_j][p_1]^{n_1}[p_2]^{n_2}[g]^r\}^{\mu_3\cdots\mu_p}$$

$$+\bar{\delta}_{r,0}\,(d+p-2+n_1+n_2)\,\{[p_1]^{n_1}[p_2]^{n_2}[g]^{r-1}\}^{\mu_3\cdots\mu_p}$$