UV divergent parts in $d=4-2\epsilon$ dimensions.

$P=2r$

$$A_{\underbrace{0\cdots 0}_{2r}}(m_0)\Big|_{\substack{UV\\ Div}} = \frac{(m_0^2)^{r+1}}{2^r(r+1)!} \times \frac{1}{\epsilon}$$

$$B_{\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n}}(p^2; m_0, m_1)\Big|_{\substack{UV\\ Div}} = \frac{(-1)^{2+n+\cancel{r}}}{2^r}\,\Gamma(\epsilon - r) \quad \frac{(-1)^{\cancel{r}}}{r!}\frac{1}{\epsilon}$$

$P=2r+n$

$$\times \int_0^1 dx\, x^n\left(p^2x^2 + (-p^2+m_1^2-m_0^2)x + m_0^2\right)^r$$

Apply multinomial theorem

$a = p^2$
$b = -p^2 + m_1^2 - m_0^2$
$c = m_0^2$

$$(ax^2+bx+c)^r = \sum_{k_1+k_2+k_3=r}\binom{r}{k_1,k_2,k_3}(ax^2)^{k_1}(bx)^{k_2}c^{k_3}$$

then

$$x^n(ax^2+bx+c)^r = \sum_{k_1+k_2+k_3=r}\binom{r}{k_1,k_2,k_3}a^{k_1}b^{k_2}c^{k_3}\; x^{2k_1+k_2+n}$$

then integrate

$$\int_0^1 dx\, x^n(ax^2+bx+c)^r = \sum_{k_1+k_2+k_3=r}\binom{r}{k_1,k_2,k_3}a^{k_1}b^{k_2}c^{k_3}\frac{1}{2k_1+k_2+n+1}$$

$$\sum_{k_1=0}^{r}\sum_{k_2=0}^{r-k_1}\binom{r}{k_1,k_2,r-k_1-k_2}a^{k_1}b^{k_2}c^{r-k_1-k_2}\frac{1}{2k_1+k_2+n+1}$$

$\therefore$

$$B_{\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n}}(p^2; m_0, m_1)\Big|_{\substack{UV\\ Div}} = \frac{(-1)^{2+n}}{2^r\, r!}\sum_{k_1+k_2+k_3=r}\binom{r}{k_1,k_2,k_3}\frac{a^{k_1}b^{k_2}c^{k_3}}{2k_1+k_2+n+1}\times\frac{1}{\epsilon}$$

$p = 2r + n_1 + n_2$

$r = \frac{1}{2}(p - n_1 - n_2)$

$$C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}(p_1^2, q^2, p_2^2; m_0, m_1, m_2)\Big|_{\substack{UV\\DIV}}$$

$$= \frac{(-1)^{3+p-r}}{2^r}\,\Gamma(\underbrace{3-\tfrac{d}{2}-r}_{1+\epsilon}) \int_0^1 dy \int_0^{1-y} dz\, y^{n_1} z^{n_2} [\Delta]^{\overbrace{\frac{d}{2}-3+r}^{-1-\epsilon}}$$

$\Delta = p_1^2 y^2 + p_2^2 z^2 + (-q^2 + p_1^2 + p_2^2) yz$
$+ (-p_1^2 + m_1^2 - m_0^2) y + (-p_2^2 + m_2^2 - m_0^2) z + m_0^2 - i\varepsilon$

UV divergent only if $r \geq 1$.

$$= \frac{(-1)^{\cancel{3}+p-\cancel{r}}}{2^r}\left[\frac{(-1)^{\cancel{r-1}}}{(r-1)!}\left(\frac{1}{\epsilon}\right)\right] \int_0^1 dy \int_0^{1-y} dz\, y^{n_1} z^{n_2} [\Delta]^{r-1}$$

$$= \frac{(-1)^p}{2^r (r-1)!}\frac{1}{\epsilon} \int_0^1 dy \int_0^{1-y} dz\, y^{n_1} z^{n_2} \left[ay^2 + bz^2 + cyz + dy + ez + f\right]^{r-1}$$

Apply multinomial theorem:

$$(ay^2 + \ldots + f)^{r-1} = \sum_{k_1+\ldots+k_6 = r-1} \binom{r-1}{k_1, \ldots, k_6} (ay^2)^{k_1} (bz^2)^{k_2} (cyz)^{k_3} (dy)^{k_4} (ez)^{k_5} f^{k_6}$$

then

$$y^{n_1} z^{n_2} (ay^2 + \ldots + f)^{r-1} = \sum \binom{r-1}{k_1, \ldots, k_6} a^{k_1} b^{k_2} c^{k_3} d^{k_4} e^{k_5} f^{k_6}\, y^{2k_1 + k_3 + k_4 + n_1}\, z^{2k_2 + k_3 + k_5 + n_2}$$

Then integrate

$$\int_0^1 dy \int_0^{1-y} dz\, y^{p_1} z^{p_2} = \frac{p_1!\, p_2!}{(p_1 + p_2 + 2)!}$$

$\therefore$

$$C_{0\cdots0 1\cdots1 2\cdots2}\Big|_{\substack{UV\\DIV}} = \frac{(-1)^{\overset{\text{drop}}{p = n_1+n_2+2r}}}{2^r (r-1)!} \sum_{k_1+\ldots+k_6 = r-1} \binom{r-1}{k_1 \ldots k_6} a^{k_1} b^{k_2} c^{k_3} d^{k_4} e^{k_5} f^{k_6} \frac{(2k_1 + k_3 + k_4 + n_1)!\,(2k_2 + k_3 + k_5 + n_2)!}{(2k_1 + 2k_2 + 2k_3 + k_4 + k_5 + n_1 + n_2 + 2)!}\frac{1}{\epsilon}$$

✓

$$D_{\underbrace{0\cdots0}_{2r}\,\underbrace{1\cdots1}_{n_1}\,\underbrace{2\cdots2}_{n_2}\,\underbrace{3\cdots3}_{n_3}}\Bigg|_{\substack{UV\\DIV}} = \mu^{2\epsilon} e^{\gamma_E \epsilon} \frac{(-1)^{4+P-r}}{2^r}\,\Gamma(\overbrace{4-\tfrac{d}{2}}^{2+\epsilon}-r)$$

$$\times \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \; x^{n_1} y^{n_2} z^{n_3} \; \Delta(x,y,z)^{\overbrace{\frac{d}{2}-4}^{-2-\epsilon}+r}$$

$$\Delta(x,y,z) = ax^2+by^2+cz^2+dxy+exz+fyz$$
$$gx+hy+iz+j$$

$$\begin{aligned}
a &= p_1^2 && = s_1\\
b &= p_2^2 && = s_{12}\\
c &= p_3^2 && = s_4\\
d &= 2p_1\cdot p_2 && = (s_{12}+s_1-s_2)\\
e &= 2p_1\cdot p_3 && = (-s_{23}+s_1+s_4)\\
f &= 2p_2\cdot p_3 && = (s_{12}+s_4-s_3)\\
g &= (-p_1^2+m_1^2-m_0^2) && = (-s_1+m_1^2-m_0^2)\\
h &= (-p_2^2+m_2^2-m_0^2) && = (-s_{12}+m_2^2-m_0^2)\\
i &= (-p_3^2+m_3^2-m_0^2) && = (-s_4+m_3^2-m_0^2)\\
j &= m_0^2-i\varepsilon && = m_0^2-i\varepsilon
\end{aligned}$$

UV divergent only if $r \geq 2$.

Retain $\frac{1}{\epsilon}$ from $\Gamma(2+\epsilon-r) \simeq \frac{(-1)^{r-2}}{(r-2)!}\frac{1}{\epsilon} + \ldots$

$$= \frac{(-1)^{\cancel{4}+n_1+n_2+n_3+\cancel{r}}}{2^r}\frac{(-1)^{\cancel{r}-\cancel{2}}}{(r-2)!}\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \; x^{n_1}y^{n_2}z^{n_3}\;\Delta(x,y,z)^{-2+r}$$

If $r \geq 2$, $\Delta(x,y,z)^{-2+r}$ is polynomial

Apply multinomial theorem

$$\Delta(x,y,z)^{r-2} = \sum_{k_1+\cdots+k_{10}=r-2}\left[\binom{r-2}{k_1,\ldots,k_{10}}(ax^2)^{k_1}(by^2)^{k_2}(cz^2)^{k_3}(dxy)^{k_4}(exz)^{k_5}\right.$$
$$\left.\times(fyz)^{k_6}(gx)^{k_7}(hy)^{k_8}(iz)^{k_9}(j)^{k_{10}}\right]$$

then

$$x^{n_1}y^{n_2}z^{n_3}\,\Delta(x,y,z)^{r-2} = \sum_{k_1+\dots+k_{10}=r-2}\left[\binom{r-2}{k_1,\dots,k_{10}} a^{k_1}b^{k_2}\cdots i^{k_9}j^{k_{10}} \times x^{2k_1+k_4+k_5+k_7+n_1}\, y^{2k_2+k_4+k_6+k_8+n_2}\, z^{2k_3+k_5+k_6+k_9+n_3}\right]$$

Then integrate:

$$\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}dz\; x^{p_1}y^{p_2}z^{p_2} = \frac{p_1!\,p_2!\,p_3!}{(p_1+p_2+p_3+3)!}$$

$\therefore$

$$D_{0\text{-}01\text{-}12\text{-}23\text{-}3}\Big|_{\substack{UV\\ DIV}} = \frac{(-1)^{n_1+n_2+n_3}}{2^r(r-2)!}\sum_{k_1+\dots+k_{10}=r-2}\left[\binom{r-2}{k_1,\dots,k_{10}} a^{k_1}b^{k_2}\cdots i^{k_9}j^{k_n} \frac{(2k_1+k_4+k_5+k_7+n_1)!\,(2k_2+k_4+k_6+k_8+n_2)!\,(2k_3+k_5+k_6+k_9+n_3)!}{(2k_1+2k_2+2k_3+2k_4+2k_5+2k_6+k_7+k_8+k_9+n_1+n_2+n_3+3)!}\right]$$

Also $k_1+k_2+\dots+k_{10}=r-2$

$$\therefore \text{Denom}! = (\underbrace{k_1+k_2+k_3+k_4+k_5+k_6}+r-2-k_{10}+n_1+n_2+n_3+3)!$$

$$= (r-2-k_7-k_8-k_9-k_{10}+r-2-k_{10}+n_1+n_2+n_3+3)!$$

$$= (n_1+n_2+n_3+2r-1-k_7-k_8-k_9-2k_{10})!$$