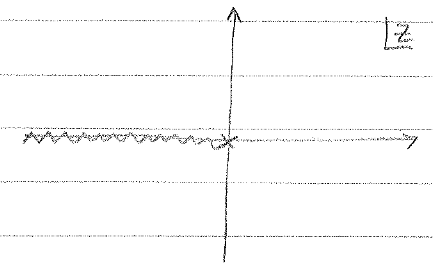
Logarithm and Veltman step ('t Hooft & Veltman)

CONVENTION: $\ln(z)$ has branch cut along negative real axis



$$\ln(x+i\epsilon) = \ln|x| + i\pi \quad \text{if } x<0.$$

$$\text{Disc}\,\ln(x) := \lim_{\epsilon\to 0^+}\left[\ln(x+i\epsilon) - \ln(x-i\epsilon)\right]$$
$$= 2\pi i \quad \text{(real part equal on both sides)}$$

$$\ln(ab) = \ln a + \ln b + \underbrace{\eta(a,b)}_{\substack{\text{Veltmann cut-crossing}\\ \text{constant}}} \Rightarrow \ln\left(a\tfrac{1}{b}\right) = \ln a - \ln b + \eta\left(a, \tfrac{1}{b}\right)$$

$$\eta(a,b) = \begin{cases} +2\pi i, & \arg(a)+\arg(b) \le -\pi \\ -2\pi i, & \arg(a)+\arg(b) > \pi \\ 0, & \text{otherwise} \end{cases}$$

Special cases:

① $\ln(ab) = \ln a + \ln b$ if $\text{Im}(a)$ and $\text{Im}(b)$ have opp. signs.

② $\ln\left(a\frac{1}{b}\right) = \ln a - \ln b$ if " " have same signs.

$$\left[\begin{array}{l} \ln a + \ln a^* = \ln(a a^*) \\ \ln a - \ln a^* = 2\ln\left(a\frac{1}{|a|}\right) \neq \ln\left(\frac{a}{a^*}\right). \end{array}\right.$$

③ $\ln(ab - i\epsilon) = \ln(a - i\epsilon) + \ln\left(b - \frac{i\epsilon}{a}\right)$ for $a, b, \in \mathbb{R}$.

Also (trivial) square-root identity:

④ $(\sqrt{z})^2 = z$ but $\sqrt{(z^2)} \neq z$.

⑤ $$\int_a^b dz\, \frac{1}{z-x} = \ln\left(\frac{b-x}{a-x}\right) \quad \text{[correctly accounts for cut]}$$

$$= \ln(b-x) - \ln(a-x) + \eta\left(b-x, \frac{1}{a-x}\right)$$

Integral representation

$$\eta(a,b) = \ln(ab) - \ln a - \ln b$$

$$= \int_1^{ab} \frac{dx}{x} - \int_1^{a} \frac{dx}{x} - \int_1^{b} \frac{dx}{x}$$

In first two integrals, ch. var:

$x = az \qquad x: 1 \to ab$

$dx = a\,dz \qquad ^{(1)} z: \frac{1}{a} \to b \qquad ^{(2)} z: \frac{1}{a} \to 1$

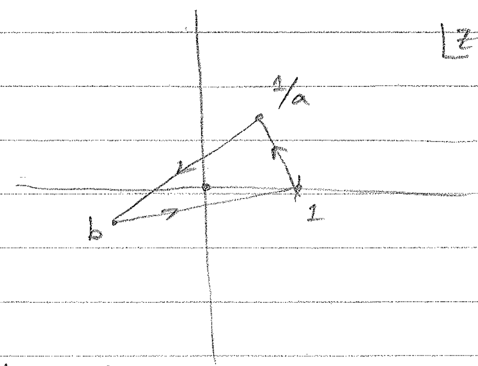
$\frac{dx}{x} = \frac{dz}{z}$

$$\eta(a,b) = \int_{\frac{1}{a}}^{b} \frac{dz}{z} \underbrace{- \int_{\frac{1}{a}}^{1} \frac{dz}{z}}_{\text{flip range of integration}} - \int_1^{b} \frac{dx}{x}$$

$$= \left(\int_{\frac{1}{a}}^{b} + \int_{b}^{1} + \int_{1}^{\frac{1}{a}} \right) \frac{dz}{z}$$

or

$$\boxed{\eta(a,b) = \int_{\Delta(\frac{1}{a},b,1)} \frac{dz}{z}}$$



Gives $2\pi i$ if $(0,0)$ is inside $\Delta\left(\frac{1}{a}, b, 1\right)$

Gives $-2\pi i$ if " " " $\Delta\left(b, \frac{1}{a}, 1\right)$

Gives 0 if " " outside

$$\eta(a,b) = \eta(b,a) = -\eta\left(\frac{1}{a}, \frac{1}{b}\right)$$

Numerical implementation of Veltman's η function — incl. inf. im. parts.

$$\eta(a,b) = \begin{cases} 2\pi i, & \mathrm{Im}(a)<0 \ \& \ \mathrm{Im}(b)<0 \ \& \ \mathrm{Im}(ab) \underset{(-)}{\geq} 0 \\ -2\pi i, & \mathrm{Im}(a)\underset{(-)}{\geq} 0 \ \& \ \mathrm{Im}(b)\underset{(-)}{\geq} 0 \ \& \ \mathrm{Im}(ab) < 0 \\ 0 \end{cases}$$

If a or b are real-valued and negative, their infinitesimal imaginary parts may be needed.

Define: $a \equiv \bar{a} + \alpha i \varepsilon$

$b = \bar{b} + \beta i \varepsilon$

$$ab = \overline{ab} + \gamma i\varepsilon \ ; \quad \gamma = \bar{a}\beta + \bar{b}\alpha$$

$$\text{Then } \mathrm{Im}(\bar{a},\alpha) = \begin{cases} \mathrm{Im}(\bar{a}) & \text{if it is non-zero} \\ \mathrm{Re}(\alpha) & \text{if } \mathrm{Im}(\bar{a})=0. \end{cases}$$

etc.

$$\eta(\bar{a},\alpha,\bar{b},\beta) = \begin{cases} 2\pi i, & \mathrm{Im}(\bar{a},\alpha)<0 \ \& \ \mathrm{Im}(\bar{b},\beta)<0 \ \& \ \mathrm{Im}(\overline{ab},\gamma) \underset{(-)}{\geq} 0 \\ -2\pi i, & \mathrm{Im}(\bar{a},\alpha)\underset{(-)}{\geq} 0 \ \& \ \mathrm{Im}(\bar{b},\beta)\underset{(-)}{\geq} 0 \ \& \ \mathrm{Im}(\overline{ab},\gamma) < 0 \\ 0, & \text{else.} \end{cases}$$