Multivariable Veltman η

The representation $\eta(z_1, z_2, \ldots) = -2\pi i \left\lfloor \frac{\arg(z_1) + \ldots + \arg(z_n) + \pi}{2\pi} \right\rfloor = 2\pi i \left\lfloor \frac{1}{2} - \frac{\arg(z_1)}{2\pi} \ldots - \frac{\arg(z_n)}{2\pi} \right\rfloor$

can be computationally expensive (because of all the arg functions)

Equate ② = ① of continued dilogarithm relations $\mathrm{Ln}(z_1, z_2, \ldots)$

$$\underline{\eta(z_1, z_2, \ldots) = \ln(z_1 z_2 \cdots) - \ln(z_1) - \ln(z_2) - \ldots}$$

$\therefore$ Veltman η is totally symmetric

Then relate multivariable Veltman η to two-variable η:

$\boxed{2}$ $\eta(z_1, z_2) := \ln(z_1 z_2) - \ln(z_1) - \ln(z_2)$

$\boxed{3}$
$$\begin{aligned}
\eta(z_1, z_2, z_3) &= \ln(z_1 z_2 z_3) - \ln(z_1) - \ln(z_2) - \ln(z_3) \\
&= \ln(z_1 z_2 z_3) + \eta(z_1, z_2) - \ln(z_1 z_2) - \ln(z_3) \\
&= \eta(z_1 z_2, z_3) + \eta(z_1, z_2)
\end{aligned}$$

$\boxed{4}$
$$\begin{aligned}
\eta(z_1, z_2, z_3, z_4) &= \ln(z_1 z_2 z_3 z_4) - \ln(z_1) - \ln(z_2) - \ln(z_3) - \ln(z_4) \\
&= \ln(z_1 z_2 z_3 z_4) + \eta(z_1, z_2, z_3) - \ln(z_1 z_2 z_3) - \ln(z_4) \\
&= \eta(z_1 z_2 z_3, z_4) + \eta(z_1, z_2, z_3). \\
&= \eta(z_1 z_2 z_3, z_4) + \eta(z_1 z_2, z_3) + \eta(z_1, z_2)
\end{aligned}$$

More generally (VELTMAN-ETA reduction formula):

$$\begin{aligned}
\eta(z_1, z_2, \ldots z_n) &= \eta(z_1 z_2 \cdots z_{n-1}, z_n) + \eta(z_1, z_2, \ldots, z_{n-1}) \\
&= \eta(z_1 z_2 \cdots z_{n-1}, z_n) + \eta(z_1 z_2 \cdots z_{n-2}, z_{n-1}) + \eta(z_1 z_2 \cdots z_{n-3}, z_{n-2}) \\
&\qquad + \ldots + \eta(z_1, z_2) \\
&= \sum_{i=1}^{n-1} \eta\left(\prod_{j=1}^{i} x_j, x_{i+2}\right)
\end{aligned}$$

Define VELTMAN index: $\eta = 2\pi i\, \nu$

$$\nu(z_1, z_2, \ldots z_n) = \sum_{i=1}^{n-1} \nu\left(\prod_{j=1}^{i} x_j, x_{i+1}\right)$$