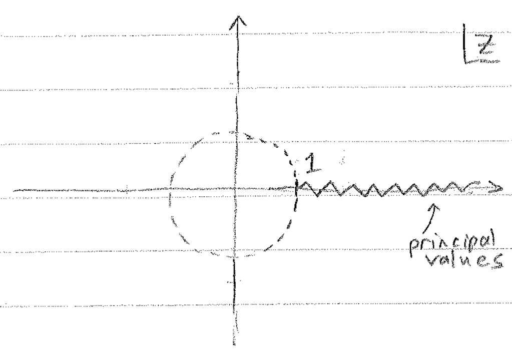
Dilogarithm (Spence function)

Definition (principal Riemann sheet)

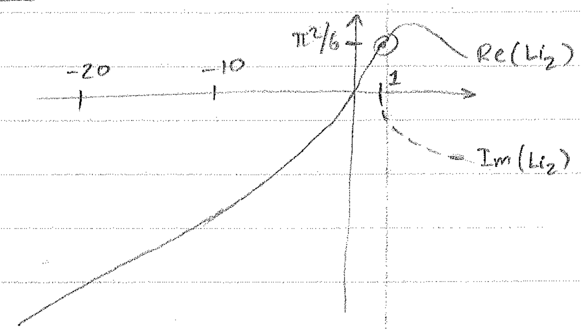
$$\mathrm{Li}_2(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^2} \equiv \int_0^z dt\, \frac{-\ln(1-t)}{t} \equiv \mathrm{Sp}(z)$$

$$\equiv \int_0^1 dt\, \frac{-\ln(1-zt)}{t}$$

Analytic structure (principal sheet)



Plot



Imaginary values on cut: $(z > 1)$

$$\mathrm{Im}\,\mathrm{Li}_2(z) \equiv \lim_{\epsilon\to 0^+} \mathrm{Im}\,\mathrm{Li}_2(z - i\epsilon)$$

$$= -\pi \ln(z)\ \theta(z-1)$$

Discontinuity across cut:

$$\mathrm{Disc}\ \mathrm{Li}_2(z) := \lim_{\epsilon\to 0^+} \left[\mathrm{Li}_2(z-i\epsilon) - \mathrm{Li}_2(z+i\epsilon) \right]$$

$$= -2\pi i\ \ln(z)\, \theta(z-1) \quad \text{(real part equal on both sides)}$$

For same function, but one that evaluates on the upper edge of cut, apply id 3 (on next page):

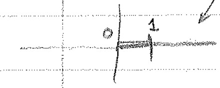
$$\lim_{\epsilon\to 0} \mathrm{Li}_2(z+i\epsilon) = -\mathrm{Li}_2\left(\frac{z}{z-1} - i\epsilon\right) - \frac{1}{2}\ln^2\left(\frac{1}{1-z} + i\epsilon\right)$$

Transformation formulae

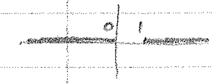
L. Maximon. The dilogarithm function for complex argument (2003)

Validity of transformation (everywhere except on line)

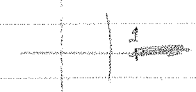
1) $\mathrm{Li}_2\left(\frac{1}{z}\right) = -\mathrm{Li}_2(z) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2(-z)$



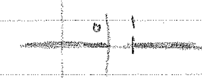
(*) 2) $\mathrm{Li}_2(1-z) = -\mathrm{Li}_2(z) + \frac{\pi^2}{6} - \ln(1-z)\ln(z)$



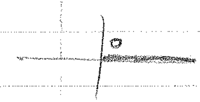
3) $\mathrm{Li}_2\left(\frac{z}{z-1}\right) = -\mathrm{Li}_2(z) - \frac{1}{2}\ln^2(1-z)$



4) $\mathrm{Li}_2\left(\frac{z-1}{z}\right) = \mathrm{Li}_2(z) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2 z + \ln(1-z)\ln(z)$



5) $\mathrm{Li}_2\left(\frac{1}{1-z}\right) = \mathrm{Li}_2(z) + \frac{\pi^2}{6} + \ln(-z)\ln(1-z) - \frac{1}{2}\ln^2(1-z)$



6) $\mathrm{Li}_2(-z) + \mathrm{Li}_2(z) = 2\,\mathrm{Li}_2(z^2)$

Proof of (*):

$$\mathrm{Li}_2(z) = \int_0^z dt\, \frac{-\ln(1-t)}{t}$$

ch. var: $t = 1-s$, $dt = -ds$; $t=0 \Rightarrow s=1$, $t=z \Rightarrow s=1-z$

$$= \int_1^{1-z} (-ds)\, \frac{-\ln(1-1+s)}{1-s} = \int_{1-z}^{1} ds\, \frac{-\ln(s)}{1-s}$$

write: $1-s = \frac{d}{ds}\ln(1-s)$

$$= \int_{1-z}^{1} ds\, \ln(s)\, \frac{d}{ds}\ln(1-s)$$

$$= \int_{1-z}^{1} ds\, \frac{-\ln(1-s)}{s} + \left[0 - \ln(1-z)\ln(z)\right]$$

$$= \left[\int_0^1 - \int_0^{1-z}\right] ds\, \frac{-\ln(1-s)}{s} - \ln(1-z)\ln(z)$$

$$= -\mathrm{Li}_2(1-z) + \underbrace{\mathrm{Li}_2(z)}_{\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{6}} - \ln(1-z)\ln(z)$$

(proof of this made Euler famous)